SHADOW TREES OF MANDELBROT SETS

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Abstract. The topology and combinatorial structure of the Mandelbrot set $\mathcal{M}^d$ (of degree $d \geq 2$) can be studied using symbolic dynamics. Each parameter is mapped to a kneading sequence, or equivalently, an internal address; but not every such sequence is realized by a parameter in $\mathcal{M}^d$. Thus the Abstract Mandelbrot set is a subspace of a larger, partially ordered symbol space, Λ^d. In this paper we find an algorithm to construct "visible trees" from symbolic sequences which works whether or not the sequence is realized. We use this procedure to find a large class of addresses that are nonrealizable, and to prove that all such trees in Λ^d actually satisfy the Translation Principle (unlike in $\mathcal{M}^d$). We also study how the existence of a hyperbolic component with a given address depends on the degree d: addresses can be sorted into families so that at least one address of each family is realized for sufficiently large d.

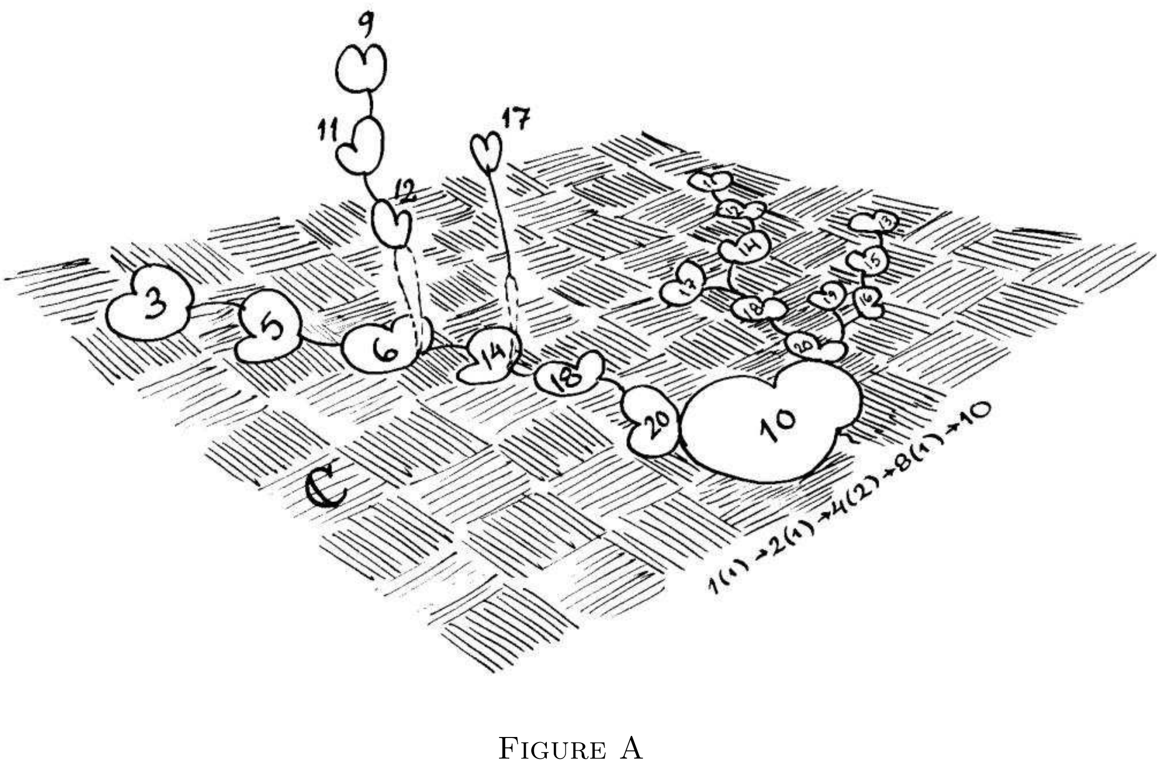

Figure A

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Introduction

The generalized Mandelbrot sets $\mathcal{M}^d$ are interesting not only from the complex dynamical and fractal geometric points of view, but also for their combinatorial properties. After some observations of the general shape and size of the sets in section §1, we focus on the combinatorics.

We use the symbolic dynamical theory developed in [D-H], [Do-1], [Thu], [Mil-2], [Ate], [B-K], [Pen], [L-S-1] (etc.) and summarize it in sections §2...§4; see also [Kel], [Do-2]. Given an angle, one finds its *kneading sequence* and translates it to an *internal address* using the *Lau–Schleicher Algorithm* (LSA). This tells where in $\partial\mathcal{M}^d$ the external ray with that angle lands at (or accumulates to). Every parameter in $\mathcal{M}^d$ is thus attached to an address $n_1(s_1) \rightarrowtail n_2(s_2) \rightarrowtail \dots$, where the *rough address* $n_1 \rightarrowtail n_2 \rightarrowtail \dots$ is a strictly increasing sequence of integers starting with $n_1 = 1$, and the sector numbers $s_i \in \mathbb{Z}_d \setminus \{0\}$ $\forall i \in \mathbb{N}$.

An interesting fact is that not every such sequence occurs as internal addresses of parameters in $\mathcal{M}^d$. This raises the fundamental question: which formal addresses are realizable and which are not? We give some partial answers for that. A characterization in the quadratic case ($d = 2$) is being done by Bruin and Schleicher [B-S]; they use Hubbard trees to study the dynamics determined by the address, whereas our arguments are mainly based on the Structure Theorems, i.e., knowledge about landing of rational external parameter rays [D-H], [Mil-2], [Sch-2], [Ebe], [Rie].

Denoting by $\simeq$ the equivalence relation that identifies parameters sharing an internal address, the quotient space $\mathcal{M}^d/\simeq$ is actually a proper subset of a larger symbol space Λ^d of *formal addresses*, which we study in section §5. Using the fact that LSA is well defined regardless of the existence of a component with a given internal address, we extend the combinatorial tree structure of the Mandelbrot set into the space Λ^d.

The internal address (or equivalently, kneading sequence) not only tells which way to walk from the origin in order to reach a parameter with a given external angle [L-S-1]; we find that it also tells which components there are ahead if one continues away from the origin. In §6 we develop a new algorithm (based on LSA) to construct the tree of formal addresses that are *visible* from a given base. The hyperbolic components realizing them in $\mathcal{M}^d$ – if such exist! – are then visible from the base sector in the sense of [L-S-1]. Moreover, our tree-growing algorithm can be used to construct *shadow trees* which only contain non-realizable formal addresses, or "non-existent" components. Finding shadow trees within formal trees thus helps to distinguish existent components from nonexistent ones.

Another main result of this paper is that the "Translation Principle" (3.6) of combinatorial equivalence holds for formal trees (6.14) even though it is not true in general for real visible trees. In this sense the symbol space Λ^d is more regular than the Mandelbrot set.

In §7 we use results from §6 to prove a "folklore theorem" about wake-widths of hyperbolic satellite components. This result is needed in §8, as we study how the realizability of formal addresses depends on the degree d. We find that every rough address is realized with some set of sector numbers, for a sufficiently large d depending on the length of the address.

The non-existent parameters thus form a "shadow space" with similar tree structures as the real one. Figure A shows an example of a ten-periodic hyperbolic component in $\mathcal{M}_3$ with formal trees stemming from its two 1/2-satellites. In contrast to the natural world, here the actual visible trees lie flat on the complex plane while the shadow trees are sticking up to another dimension.

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§1 Geometric properties of $\mathcal{M}^d$

The *Mandelbrot set of degree* d is defined as the set of parameters $c \in \mathbb{C}$ for which the Julia set of the polynomial $P_c : z \mapsto z^d + c$ is connected; or equivalently,

$$\mathcal{M}^d := \big\{ c \in \mathbb{C} : P_c^n(0) := P_c \circ \cdots \circ P_c(0) \nrightarrow \infty \text{ as } \; n \to \infty \big\}. \tag{1.1}$$

(By a classical result of Fatou and Julia, the Julia set of a polynomial is connected if and only if all its finite critical points have bounded orbits; zero is the only finite critical point of P_c [C-G, III], [Mil-1].)

1.2 Hyperbolic components and sectors. The *multiplier* of a k-periodic orbit $\{z_1, \dots, z_k\}$ (where $z_{j+1} = z_j^d + c$ and $j \in \mathbb{Z}_k$) of P_c is

$$\lambda(c) := \big(P_c^k\big)'(z_1) = P_c'(z_k) \cdot \ldots \cdot P_c'(z_1) = d^k \cdot (z_1 \cdot \ldots \cdot z_k)^{d-1}.$$

The orbit is attracting if $|\lambda| < 1$, neutral if $|\lambda| = 1$ (in particular, parabolic if $\lambda = e^{i2\pi\phi}$ with ϕ in $\mathbb{Q}$), and repelling if $|\lambda| > 1$. For every $k \in \mathbb{N}$, the set

$$\mathcal{H}^d(k) := \{ c \in \mathbb{C} : P_c \text{ has an attracting } \; k\text{-periodic orbit} \}$$

is well known to be an open subset of $\mathcal{M}^d$, and its connected components are called the *hyperbolic components*. Each polynomial P_c with c in some hyperbolic component $\mathcal{H} \subset \mathcal{H}^d(k)$ has a unique k-periodic orbit whose multiplier has absolute value less than one; that is, $\lambda(c)$ belongs to the unit disk $\mathbb{D}$. The number k is also called the *period* of the component $\mathcal{H}$, denoted by $\langle \mathcal{H} \rangle$.

The Douady-Hubbard-Sullivan Theorem [C-G], [Mil-1] is generalized [Ebe] to the case $d \geq 2$ as follows: the multiplier map $\lambda_{\mathcal{H}} : \mathcal{H} \to \mathbb{D}$ is well-defined for every $\mathcal{H}$, and it is an analytic $(d-1)$-to-one mapping which extends continuously to the boundary.

For each angle[1] $\phi \in \mathbb{R}/\mathbb{Z}$, the $d-1$ preimages under $\lambda_{\mathcal{H}}$ of the rays $\{ re^{i2\pi\phi} : 0 < r \leq 1\}$ are called the *internal rays* of $\mathcal{H}$ with arguments $(\phi + n)/(d-1)$, $n = 0, \dots, d-2$. In particular, the internal rays mapped by $\lambda_{\mathcal{H}}$ to the positive real axis (that is, $\phi = 0$) divide the hyperbolic component $\mathcal{H}$ into $d-1$ *sectors*.

Polynomials P_c with $c \in \partial\mathcal{H}$ at the limit points of these $d-1$ sector-dividing rays thus have a parabolic orbit with multiplier $\lambda_{\mathcal{H}}(c) = 1$. They are called the *root*

[1] Throughout this paper, all angles are measured in full turns (not radians or degrees).

and *co-roots* of $\mathcal{H}$ (see 2.8 below). The branching point $\lambda_{\mathcal{H}}^{-1}(0)$ is the *centerpoint* of $\mathcal{H}$.

The boundary of each hyperbolic sector also contains parameters $c = \lambda_{\mathcal{H}}^{-1}(e^{i2\pi p/q})$ with any rational number $p/q \in]0,1[$. Any neighborhood of such a point c contains parameters $\tilde{c}$ whose polynomials have attracting cycles with period $q\langle\mathcal{H}\rangle$. Hence $\tilde{c}$ belongs to another hyperbolic component $\mathcal{K}$ whose root point is c. Then $\mathcal{K}$ is the *p/q-satellite* of $\mathcal{H}$.

Hyperbolic components whose roots do not belong to the boundary of another sector are called *primitive* components. Figure B shows all hyperbolic components with periods less than five in $\mathcal{M}^3$. Notice the satellites of the two sectors of the main component with internal angles $1/4, 1/3, 1/2, 2/3, 3/4$. There are also four primitive components with period three, each connected to a four-periodic satellite of a two-periodic sector by a "thread".

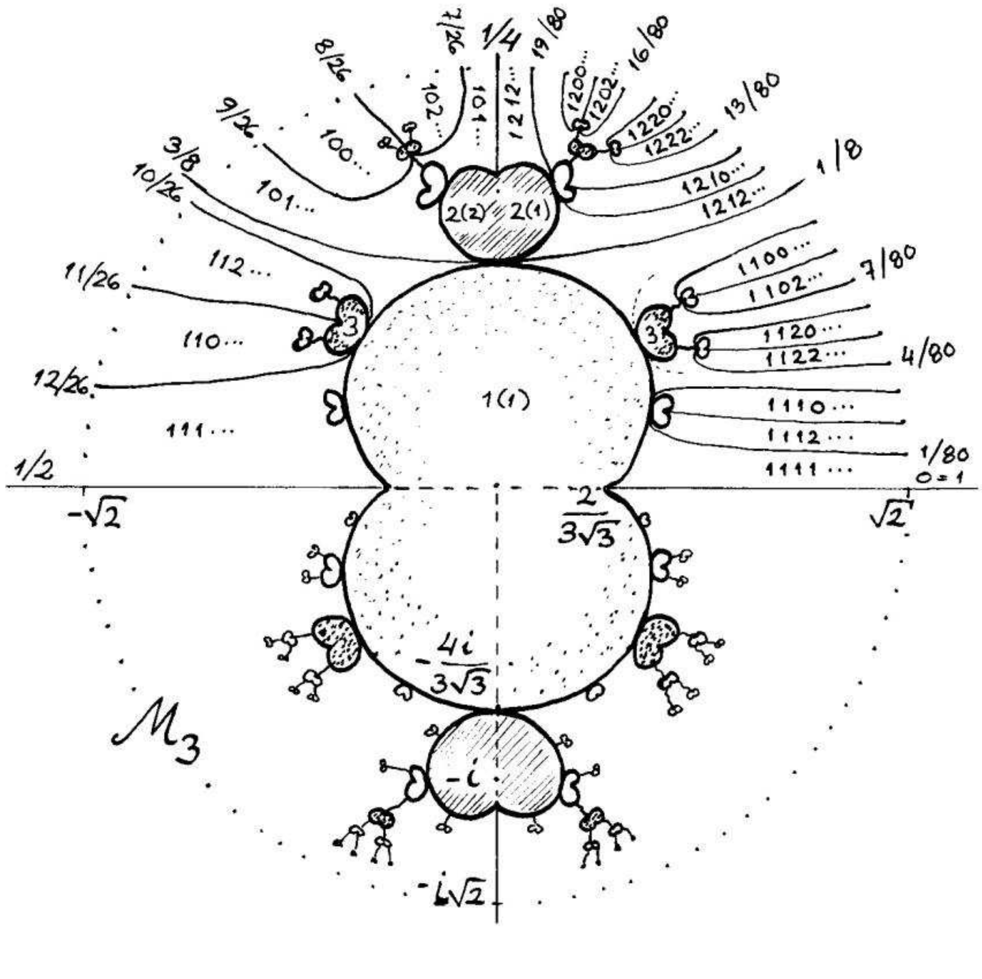

Figure B

1.3 The main epicycloid. As an example, we consider the simplest special case $k = 1$. There is a homeomorphism γ between the closure of the hyperbolic component $\mathcal{H} = \mathcal{H}^d(1)$ and the set of corresponding attracting fixed points, which is a disk.

If $x \in \mathbb{C}$ is the fixed point of the polynomial $P_c : z \mapsto z^d + c$, then $c = x - x^d$.

Writing $x = |x|e^{i2\pi\phi}$, we get

$$\gamma(|x|, \phi) = |x|e^{i2\pi\phi} - |x|^d e^{i2\pi\phi d}. \tag{1.4}$$

Fixing one of the parameters (angle ϕ or modulus $|x|$), this yields a family of curves; $\gamma_\phi :]0, \infty[\to \mathbb{C}$ or $\gamma_{|x|} : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$, respectively. Now, the arc $\gamma_\phi|_{]0,1[}$ is an internal ray for each angle ϕ.

The orbit $\{x\}$ has multiplier $\lambda = P_c'(x) = dx^{d-1}$, so $|\lambda| = d|x|^{d-1}$, and $x \mapsto \lambda$ is an analytic $(d-1)$-to-one mapping. For a fixed modulus $|x| = a$, (1.4) gives a classical cycloidal curve [L-S-2] called an *epitrokhoid* [Kah]; γ_a is drawn by a point p on a circle with radius $b = a^d$ spinning at constant angular velocity $d \cdot v$ while its centerpoint is moving at angular velocity v along a circle of radius $a = |x|$ centered at origin. This curve intersects itself exactly if the point p changes direction on its orbit seen from the origin[2] ; that is, if

$$av - bdv < 0 \iff d > \frac{a}{b} = |x|^{1-d} = \frac{d}{|\lambda|} \iff |\lambda| > 1.$$

The mapping $x \mapsto c$ is thus a homeomorphism when $|\lambda| \le 1$, or $|x| \le d^{-1/(d-1)}$. In other words, the disk of attracting and neutral fixed points x maps homeomorphically to the closure of the hyperbolic component. In the limiting case, $|x| = d^{-1/(d-1)}$, the velocity $(a - bd)v$ of the point p is zero when p is nearest to the origin, so the curve $\gamma_{|x|}(\phi)$ has a sharp cusp inwards when $(d-1)\phi = 0$ mod 1. This curve is an *epicycloid*, and the cusp points are the root and the co-roots.

1.5 Remark. For every d, the symmetry group of $\mathcal{M}^d$ is the dihedral group D_{d-1}. That the Mandelbrot set is symmetric with respect to reflection and rotation by an angle $1/(d-1)$, follows from the fact that the polynomials $P_{\overline{c}}$ and $P_{\rho c}$ (where $\rho = e^{i2\pi/(d-1)}$) have dynamics similar to that of P_c. For details and the converse statement (there are no other symmetries), see [L-S-2] or [S-S].

To finish the first section, we prove another "folklore" result concerning the size of the Mandelbrot set of degree $d \ge 2$:

1.6 Proposition.

1. $c \in \mathcal{M}^d \implies |c| \le 2^{1/(d-1)}$.
2. $c \in \mathcal{M}^d \iff |P_c^n(0)| \le 2^{1/(d-1)} \quad \forall n \in \mathbb{N}$.
3. *The bound $2^{1/(d-1)}$ for $\mathcal{M}^d$ is sharp for even values of d but not for odd ones.*

Proof. (1) Denote the iteration sequence by $(z_n)_{n\in\mathbb{N}}$, such that $z_0 = 0$ and $z_n = z_{n-1}^d + c$. Take an arbitrary $\varepsilon > 0$ and assume that $|c| \ge 2^{1/(d-1)} + \varepsilon$; we show that then $c \notin \mathcal{M}^d$. The assumption first implies that

$$|c|^{d-1} \ge 2 + (d-1)2^{(d-2)/(d-1)}\varepsilon + \cdots + \varepsilon^{d-1} > 2 + (d-1)\varepsilon \quad \text{and}$$
$$|z_2| = |c^d + c| \ge |c|(|c|^{d-1} - 1) > |c|(1 + (d-1)\varepsilon).$$

[2]This is why epitrokhoids were used as a model of planetary motions before Kepler's invention of elliptic orbits; viewed from the Earth, e.g. Mars seems to make backward loops against the background stars.

Assuming inductively that $|z_n| > |c|(1 + (d-1)\varepsilon)^{n-1}$ for some $n \in \mathbb{N}$, we obtain

$$\begin{aligned} |z_{n+1}| &= |z_n^d + c| \geq |z_n|^d - |c| > |c|^d \big(1 + (d-1)\varepsilon\big)^{(n-1)d} - |c| \\ &= |c| \Big(|c|^{d-1} \big(1 + (d-1)\varepsilon\big)^{(n-1)d} - 1 \Big) \\ &> |c| \Big(\big(1 + (d-1)\varepsilon\big)^{(n-1)d+1} + \big(1 + (d-1)\varepsilon\big)^{(n-1)d} - 1 \Big) \\ &> |c| \big(1 + (d-1)\varepsilon\big)^n. \end{aligned}$$

Now $|z_n| > |c|\big(1 + (d-1)\varepsilon\big)^{n-1}$ for all $n \in \mathbb{N}$, and $|z_n| \to \infty$ as $n \to \infty$. Because of (1.1), $|c|$ must be at most $2^{1/(d-1)}$ if $c \in \mathcal{M}^d$, so we have proved the first statement.

(2) The latter condition is sufficient by definition. To show that it is also necessary, assume that $|z_n| \geq 2^{1/(d-1)} + \varepsilon$ for some $c \in \mathcal{M}^d$ and $n \in \mathbb{N}$; by (1), $|c| \leq 2^{1/(d-1)}$. Then

$$\begin{aligned} |z_{n+1}| = |z_n^d + c| \geq |z_n|^d - |c| > 2^{d/(d-1)} + d\varepsilon - 2^{1/(d-1)} = 2^{1/(d-1)} + d\varepsilon \\ \implies |z_{n+k}| > 2^{1/(d-1)} + d^k \varepsilon \ \forall k \in \mathbb{N}, \end{aligned}$$

so the sequence is unbounded, which contradicts the assumption that $c \in \mathcal{M}^d$.

(3) If $c = -2^{1/(d-1)}$, then 0 is mapped to a fixed point in two iterations:

$$P_c : 0 \mapsto -2^{1/(d-1)} \mapsto (-2^{1/(d-1)})^d - 2^{1/(d-1)} = 2^{1/(d-1)} \mapsto 2^{1/(d-1)}.$$

Hence $-2^{1/(d-1)} \in \mathcal{M}^d$. By dihedrality (1.5), also the points $2^{1/(d-1)} e^{i\pi(1+2k)/(d-1)}$ belong to $\mathcal{M}^d$ for every $k = 0, \ldots, d-2$.

For *odd* values of d however, this bound is not sharp. If $c = 2^{1/(d-1)} e^{i2\pi\varphi}$ for any φ, then

$$P_c^2(0) = 2^{1/(d-1)} e^{i2\pi\varphi} (2e^{i2\pi\varphi(d-1)} + 1).$$

This number has absolute value greater than $2^{1/(d-1)}$ unless φ is an odd multiple of $1/2(d-1)$. But then $P_c^2(0) = -2^{1/(d-1)} e^{i2\pi\varphi}$, so

$$\big|P_c^3(0)\big| = \big|2^{1/(d-1)} e^{i2\pi\varphi} (-2e^{i2\pi\varphi(d-1)} + 1)\big| = 3 \cdot 2^{1/(d-1)} > 2^{1/(d-1)};$$

by (2), c must not be in $\mathcal{M}^d$. $\square$

(Since $\mathcal{M}^d$ is compact, there must be some number $\delta_d < 2^{1/(d-1)}$ such that $\mathcal{M}^d \subset \overline{B}(0, \delta_d)$ when d is odd.) From 1.3 and 1.6 we conclude:

$$\overline{B}\big(0, (d-1)d^{-d/(d-1)}\big) \subset \overline{\mathcal{H}}^d(1) \subset \mathcal{M}^d \subset \overline{B}(0, 2^{1/(d-1)}).$$

Since the radii of both these bounding disks tend to 1 as d tends to infinity, the main epicycloid becomes increasingly disk-like and dominating in comparison to the area of the whole set $\mathcal{M}^d$.

§2 Rays, orbit portraits, and kneading sequences

This section presents some important combinatorial tools that have been used for determining the structure of the Mandelbrot sets from the dynamics of the polynomials with different parameters.

2.1 External rays. The *(dynamical) external ray* with angle θ of the filled Julia set K_c of a polynomial $P_c : z \mapsto z^d + c$, denoted by $\mathcal{R}^c_\theta$, is the preimage of the radial line $\{re^{i2\pi\theta} : r > 1\}$ under the conformal *Böttcher map* from $\widehat{\mathbb{C}} \setminus K_c$ to the exterior of the closed unit disk; consult [C-G] or [Bea] for details. The *parameter rays* $\mathcal{R}^{\mathcal{M}^d}_\theta$ are defined similarly as preimages of straight rays under the conformal mapping obtained by evaluating the Böttcher map in c itself. If an external ray $\mathcal{R}_\theta$ lands (that is, has a limit when $r \searrow 1$) at a point z, then θ is called the external angle of z.

2.2 Periodic angles. The combinatorial approach to $\mathcal{M}^d$ is based on the fact that the polynomial P_c maps each dynamical ray to another ray multiplying its argument by d. An angle θ in the circle $\mathbb{R}/\mathbb{Z}$ is *k-periodic* under the d-tupling map $\sigma_d : \theta \mapsto d\theta \mod 1$ if and only if it is rational and of the form $\theta = t/(d^k - 1)$ for some integers t and (minimal) k. For example (see Figure C),

$$\sigma_3 : \frac{3}{11} \mapsto \frac{9}{11} \mapsto \left(\frac{27}{11} =\right) \frac{5}{11} \mapsto \frac{4}{11} \mapsto \frac{1}{11} \mapsto \frac{3}{11},$$

so these angles are five-periodic under tripling; note that $3/11 = 66/(3^5 - 1)$.

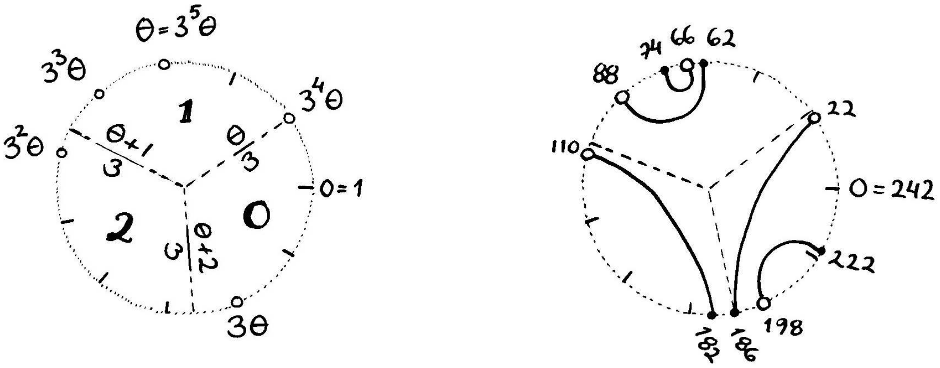

FIGURE C

2.3 Orbit portraits. This theory was developed by Milnor; see [Mil-2] for details and proofs in the quadratic case and [Ebe], [Rie] for generalizations to the case $d \geq 2$.

If some dynamic rays with rational angles land at a parabolic or repelling k-periodic orbit $\{z_1, \ldots, z_k\}$ where $z_{j+1} = z_j^d + c$, and A_j consists of all external angles of the point z_j for each $j \in \mathbb{Z}_k$, then the following holds:

(1) Every A_j has the same finite number, v, of angles.
(2) The d-tupling modulo 1 map σ_d takes A_j to A_{j+1} bijectively, preserving the cyclic order of the angles.
(3) Every angle in $A_1 \cup \cdots \cup A_k$ is periodic with the same period, qk with some $q \in \mathbb{N}$.
(4) The sets A_j are pairwise unlinked (i.e., are contained in disjoint intervals of the circle).

The set $\Theta = \{A_1, \ldots, A_k\}$ is then called the *orbit portrait* of the orbit in question. Moreover, every set satisfying the conditions (1) ... (4) above, occurs as the portrait of an orbit of some polynomials P_c.

An orbit portrait is *essential* if $v \geq 2$ or if $\Theta = \{\{0\}\}$. A portrait is *primitive* if $q = 1$ and *satellite* otherwise. Satellite portraits always have just one cycle of angles (so $v = q \geq 2$), whereas essential primitive portraits have two different cycles (so $v = 2$).

The connected components of each $\mathbb{R}/\mathbb{Z} \setminus A_j$ are called the complementary intervals. The *characteristic interval* $]\theta_-, \theta_+[$ of Θ is the unique shortest one of them.

The map σ^k permutes the angles $\{\theta_1, \dots, \theta_q\}$ $(0 < \theta_1 < \dots < \theta_q < 1)$ in each A_j so that for all $i \in \mathbb{Z}_q$, $\sigma^k(\theta_i) = \theta_{i+p}$ for some $p \in \mathbb{Z}_q$. The *rotation number* $p/q \in \mathbb{Q}/\mathbb{Z}$ of Θ is zero for primitive orbit portraits, and $0 < p/q < 1$ for satellite ones.

Figure C shows a primitive orbit portrait with characteristic interval $\left]\frac{66}{242}, \frac{74}{242}\right[$, degree $d = 3$, orbit period $k = 5$, $q = 1$, $v = 2$.

2.4 Kneading sequences of angles. The *kneading sequence* of an angle $\theta \in \mathbb{R}/\mathbb{Z}$,

$$K_d(\theta) = a_1 a_2 \ldots \in \{\mathbf{0}, \mathbf{1}, \dots, \mathbf{d-1}, \, {}^{\mathbf{1}}_{\mathbf{0}}, \, {}^{\mathbf{2}}_{\mathbf{1}}, \dots, \, {}^{\mathbf{0}}_{\mathbf{d-1}}\}^{\mathbb{N}},$$

is defined (according to [Ate], [B-K], [L-S-1], [Pen] etc.) as follows: The d preimages $\frac{\theta+j}{d}$ $(j \in \mathbb{Z}_d)$ of θ under the d-tupling map σ_d (2.2) divide the circle into d sectors of equal size. One of them contains $0 = 1$ and is labeled $\mathbf{0}$, the others are labeled counter-clockwise $\mathbf{1}$ through $\mathbf{d-1}$. We list the labels of sectors where iterates of θ go to:

$$K_d(\theta)_n := \begin{cases} \mathbf{j} \text{ if } \sigma_d^{n-1}(\theta) \in V_j^\theta := \left] \dfrac{\theta+j-1}{d}, \dfrac{\theta+j}{d} \right[\\ {}^{\mathbf{j+1}}_{\mathbf{j}} \text{ if } \sigma_d^{n-1}(\theta) = \dfrac{\theta+j}{d} \end{cases} \tag{2.5}$$

In the previous example (2.2 and Figure C), $K_3(3/11) = \overline{\mathbf{1021}{}^{\mathbf{1}}_{\mathbf{0}}}$. Kneading sequences never start with $\mathbf{0}$, because $\theta/d < \theta < (\theta + d - 1)/d$ for all $0 < \theta < 1$. The following result is a generalization of the quadratic case [L-S-1, 3.3], [Kau, 3.4]. The idea is shown in Figure D.

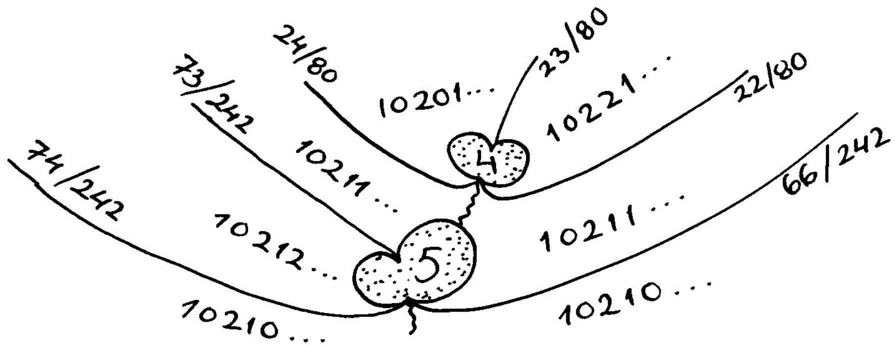

FIGURE D

2.6 Lemma. *When the angle θ moves counter-clockwise around the circle, the n^{th} entry in its kneading sequence changes from j to $j+1$ precisely when θ crosses a rational angle of the form $(rd + j)/(d^n - 1)$.*

Proof. The digit $K_d(\theta)_n$ must change exactly at such angles, because by definition,

$$K_d(\theta)_n = \,^{\mathbf{j+1}}_{\mathbf{j}} \iff d^{n-1}(\theta) = \frac{\theta+j}{d} \mod 1 \iff d^n\theta = \theta + j \mod d$$
$$\iff d^n\theta - \theta = rd + j \iff \theta = \frac{rd+j}{d^n - 1}$$

for any $r \in \{0, 1, \ldots d^{n-2}\}$. Whenever $0 < \varepsilon < 1/(d^n - 1)$,

$$\sigma_d^{n-1}(\theta + \varepsilon) = \frac{\theta + j}{d} + \frac{d^n\varepsilon}{d} = \frac{(\theta+\varepsilon) + j}{d} + \frac{(d^n - 1)\varepsilon}{d} \in V_{j+1}^{\theta+\varepsilon}$$

(because the latter term is strictly between 0 and $1/d$), so $K_d(\theta + \varepsilon)_n = \mathbf{j+1}$. Similarly, $K_d(\theta - \varepsilon)_n = \mathbf{j}$. $\square$

The two limit sequences

$$K_d^{\pm}(\theta) := \lim_{\varepsilon \to 0} K_d(\theta \pm \varepsilon) \in \{\mathbf{0}, \mathbf{1}, \ldots, \mathbf{d-1}\}^{\mathbb{N}} \tag{2.7}$$

exist for every θ. For non-periodic angles they are equal; if θ is k-periodic, they differ exactly at indices nk for every $n \in \mathbb{N}$ so that $K_d^+(\theta)_{nk} = K_d^-(\theta)_{nk} + 1$. Periodic angles have periodic limit sequences, but an n-periodic sequence may belong to an angle whose period is a multiple of n or even to a non-periodic angle.

Orbit portraits and kneading sequences are used to prove the pattern in which the external rays land at the boundary of $\mathcal{M}^d$ (see again Figure B):

2.8 Structure Theorems. [D-H], [Mil-2], [Sch-2], [Ebe]

(1) *Every parameter ray with periodic angle θ lands at a parameter $\widehat{c} \in \partial\mathcal{M}^d$. The polynomial $P_{\widehat{c}}$ has a parabolic orbit with multiplier $\lambda(\widehat{c}) = 1$ and portrait Θ containing θ.*
(2) *Every parabolic parameter $\widehat{c}$ belongs to the boundary of some hyperbolic component $\mathcal{H}$.*
(3) *If the orbit portrait Θ is non-essential, then $\widehat{c}$ is a co-root of $\mathcal{H}$ and the landing point of exactly one external ray.*
(4) *If Θ is essential, then $\widehat{c}$ is the root of $\mathcal{H}$ and the landing point of exactly two external rays, $\mathcal{R}_{\theta_-}$ and $\mathcal{R}_{\theta_+}$. The arguments of these rays are the two characteristic angles of Θ.*
(5) *Every hyperbolic component has one root and $d - 2$ co-roots.*
(6) *The two parameter rays $\mathcal{R}_{\theta_-}$ and $\mathcal{R}_{\theta_+}$ landing at $\widehat{c}$ bound the wake $W(\Theta)$ away from the origin. A polynomial P_c has a repelling orbit with portrait Θ if and only if $c \in W(\Theta)$.*

The main epicycloid is exceptional in that there is actually just one external ray landing at its rootpoint. However, the statements 4 and 6 are correct because the ray $\mathcal{R}_0 = \mathcal{R}_1$ is "counted twice": the corresponding exceptional essential orbit portrait $\Theta = \{\{0\}\}$ has characteristic interval $]0, 1[$.

It follows that the unions of external and internal rays landing at the root and co-roots of each hyperbolic component with period $k \geq 2$ divide the parameter plane into d regions. One of them contains the origin, the $d - 1$ others are called the *wakes* of the respective hyperbolic sectors. In the case of the main epicycloid, the plane just gets divided into the $d - 1$ sector wakes.

§3 Visible Trees

The Structure Theorems imply that all parameters in each Mandelbrot set sit in wakes that are either nested or disjoint; this feature gives rise to a treelike structure.

3.1 Partial order. The landing feature of parameter rays described in 2.8 determines a partial ordering in the Mandelbrot set: each parameter $c \in \mathcal{M}^d$ belongs to a certain set $U(c)$ of nested wakes of hyperbolic components and sectors. Another parameter c' is *above* c, denoted $c' \succ c$, if $U(c) \subset U(c')$. Then c is *below* c', denoted $c \prec c'$.

Belonging to the same set of wakes is obviously an equivalence relation, so we define $c' \sim c$ if $U(c') = U(c)$. In particular, all parameters in the closure of a hyperbolic component except the root are equivalent, so we can also write $\mathcal{A} \prec \mathcal{B}$ if $\mathcal{A}$ and $\mathcal{B}$ are hyperbolic components or sectors and $\mathcal{B}$ is in the wake of $\mathcal{A}$ (so, $\mathcal{A} \prec \mathcal{B} \iff W(\mathcal{A}) \supset W(\mathcal{B})$; note that the set $U(c)$ is not to be confused with the wake $W(c)$). From now on, we will usually talk in terms of equivalence classes.

This definition distinguishes a hyperbolic component from its root, in contrast to the *combinatorial class* [Kel, 1.14]. The set of all accumulation points of a parameter ray with irrational angle, together with any "weird" components that may be bounded by that set, would also be one equivalence class, so we need not worry whether the Mandelbrot set is locally connected or not.

3.2 Combinatorial arc, narrowness and visibility. The collection of all sectors $\mathcal{B}$ such that $\mathcal{C} \prec \mathcal{B} \prec \mathcal{A}$ is called the *combinatorial arc* $]\mathcal{C}, \mathcal{A}[$. We denote $[\mathcal{C}, \mathcal{A}[\,,]\mathcal{C}, \mathcal{A}]\,, [\mathcal{C}, \mathcal{A}]$ if one or both ends are included.

A hyperbolic sector $\mathcal{C}$ is *narrow* if there are no components above it with period $\langle \mathcal{C} \rangle$ or less; i.e., $\langle \mathcal{A} \rangle > \langle \mathcal{C} \rangle$ for all $\mathcal{A} \succ \mathcal{C}$. (Equivalently, the wake of $\mathcal{C}$ has width $1/(d^{\langle \mathcal{C} \rangle} - 1)$; see §7.)

A hyperbolic component $\mathcal{A}$ above sector $\mathcal{C}$ is *visible* from $\mathcal{C}$ if there are no sectors on the combinatorial arc between them with periods $\langle \mathcal{A} \rangle$ or less; i.e., $\langle \mathcal{B} \rangle > \langle \mathcal{A} \rangle$ for all $\mathcal{B} \in]\mathcal{C}, \mathcal{A}[$.

For example, in Figure B, the four-periodic component with external angles $\frac{4}{80}$, $\frac{5}{80}$, $\frac{6}{80}$ is not visible from the main epicycloid because the three-periodic sector with external angles $\frac{1}{26}$, $\frac{2}{26}$ is "blocking the view". The four-periodic sector with external angles $\frac{12}{80}$, $\frac{19}{80}$ is not narrow because the wake contains a three-periodic component with external angles $\frac{4}{26}$, $\frac{5}{26}$, $\frac{6}{26}$.

3.3 Lavaurs' Lemma. [Lav], [L-S-1, 3.8] *If $\mathcal{A} \prec \mathcal{C}$ and $\langle \mathcal{A} \rangle = \langle \mathcal{C} \rangle = k$, then there is a hyperbolic component $\mathcal{B}$ such that $\mathcal{A} \prec \mathcal{B} \prec \mathcal{C}$ and $\langle \mathcal{B} \rangle < k$.* $\square$

3.4 Trees. The set of hyperbolic components visible from a *base* sector $\mathcal{C}$ with period k are arranged in *visible trees*: every satellite component is visible from its parent sector but invisible from all other components, so each visible tree of $\mathcal{C}$ consists of a satellite component, *stem*, and a finite number of primitive components. (Satellite and primitive components were defined in 1.2.) The tree stemming from the p/q-satellite of $\mathcal{C}$ is denoted by $\mathcal{T}_{p/q}(\mathcal{C})$; all components in it must have periods at most qk.

Components $\mathcal{B} \in \mathcal{T}_{p/q}(\mathcal{C})$ that have nothing visible (from $\mathcal{C}$) above them are the *tips* of their branches. They are all narrow. The component $\mathcal{A}$ with the smallest

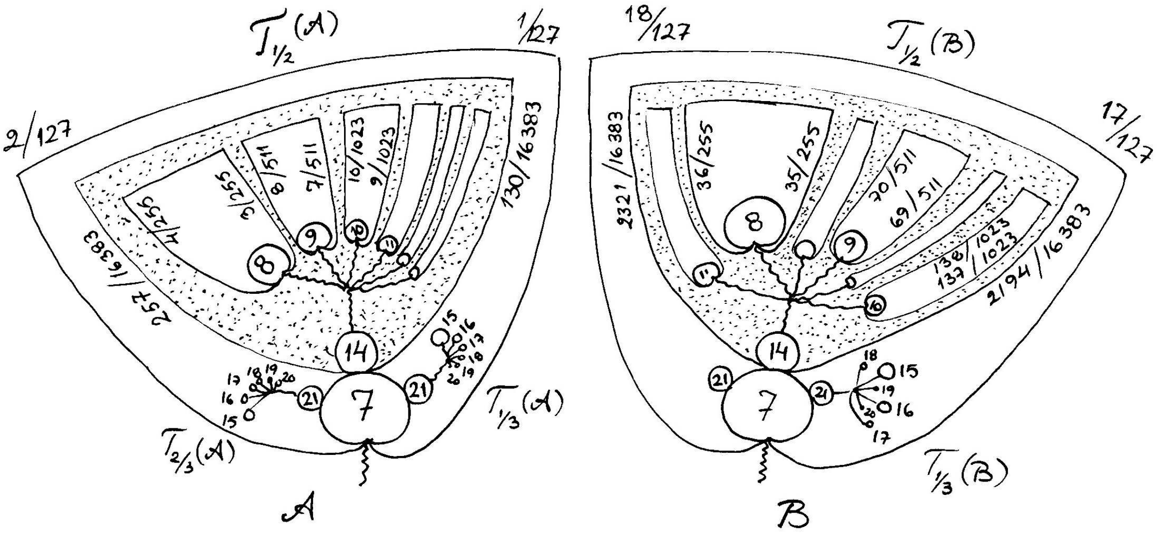

Figure E

period of all in the tree is called the *treetop.* The arc $]\mathcal{C}, \mathcal{A}]$ between the stem and the treetop is the *primary trunk* of $\mathcal{T}_{p/q}(\mathcal{C})$.

For example, Figure E shows two base components with period seven, their satellites at internal angles $1/3, 1/2, 2/3$, and six primitive components above each satellite that are visible from the base.

3.5 Equivalence of trees. Two trees $\mathcal{T}$ and $\mathcal{T}'$ are defined *combinatorially equivalent*, denoted by $\mathcal{T} \sim_c \mathcal{T}'$, if there is a homeomorphism between them which adds a constant $s \in \mathbb{Z}$ to the period of each component, i.e., maps each component with period n in $\mathcal{T}$ into a component with period $n+s$ in $\mathcal{T}'$. If $\mathcal{T} \sim_c \mathcal{T}'$ and the trees stem from the same base sector $\mathcal{C}$ at internal angles $p_1/q_1, p_2/q_2$ respectively, then the satellites must map into each other, so the constant $s = (q_2 - q_1)k$ where $k = \langle \mathcal{C} \rangle$.

The external rays, together with the natural order of angles on the circle, determine the embedding of the Mandelbrot set into the parameter plane. If a homeomorphism between the trees preserves their embeddings into the plane (either preserving or reversing orientation), we say that the trees are *topologically equivalent*: $\mathcal{T} \sim_t \mathcal{T}'$. Neither condition implies the other, but if the two trees are both topologically and combinatorially equivalent, we denote $\mathcal{T} \approx \mathcal{T}'$.

For example, the trees $\mathcal{T}_{1/2}(\mathcal{A})$ and $\mathcal{T}_{1/2}(\mathcal{B})$ in Figure E are combinatorially equivalent but not topologically (both trees have a bunch of six primitive components with equal periods respectively, but the branches are in different order). In addition, $\mathcal{T}_{1/3}(\mathcal{A}) \approx \mathcal{T}_{2/3}(\mathcal{A}) \approx \mathcal{T}_{1/2}(\mathcal{A})$, with an orientation-preserving homeomorphism (the periods of the components increase clockwise in both trees). Similarly, $\mathcal{T}_{1/2}(\mathcal{B}) \approx \mathcal{T}_{1/3}(\mathcal{B})$.

In Figure F, $\mathcal{T}_{1/3}(\mathcal{C}) \approx \mathcal{T}_{2/3}(\mathcal{C})$ with an orientation-reversing homeomorphism (the 15-periodic components in the two trees branch off the primary trunks to right and left, respectively). These trees are not equivalent to $\mathcal{T}_{1/2}(\mathcal{C})$ (not topologically *nor* combinatorially: it lacks a component with period $15 - 7 = 8$ altogether). Finally, the tree $\mathcal{T}_{1/2}(\mathcal{C})$ is topologically equivalent to $\mathcal{T}_{1/2}(\mathcal{D})$ but not combinatorially (both consist of six components in one chain, but the periods of the corresponding

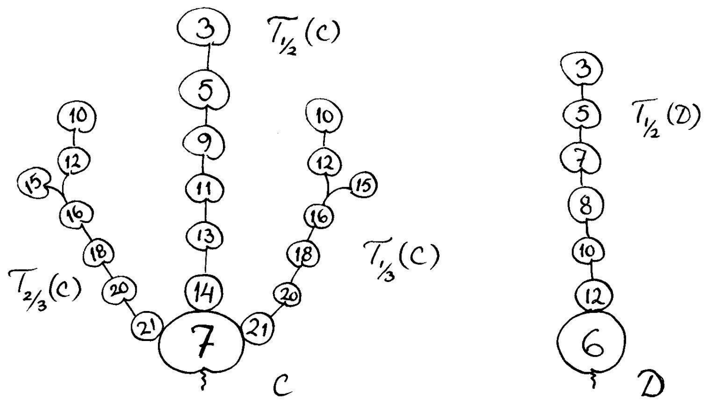

Figure F

ones do not differ by a constant number).

In this work we are more interested in combinatorial equivalence than topological, because it is relevant in terms of symbolic dynamics even for nonexistent trees (§5).

3.6 Translation Principle. *Two trees $\mathcal{T}_{p_1/q_1}$ and $\mathcal{T}_{p_2/q_2}$ of a base sector $\mathcal{C}$ are combinatorially equivalent for any p_1, p_2, q_1, and q_2.*

This statement is true for narrow base sectors; the proof in [L-S-1, 10.2] for the quadratic case $d = 2$ is generalized here for $d > 2$ in 8.2. It also holds in many other cases, but it is not true in general for visible trees of hyperbolic sectors, as shown, e.g., in [Kau,§4]. Another counter-example was shown in Figure F. However, Translation Principle does hold in general for *formal trees*, which we will prove in 6.14. The following weaker statement, *Partial Translation Principle*, is true in general at least in the quadratic case, as proved in [Kel, 3.78].

3.7 Theorem. *Let $\mathcal{C}$ be any hyperbolic sector in $\mathcal{M}^2$. Then all its visible trees $\mathcal{T}_{p/q}$, except perhaps $\mathcal{T}_{1/2}$, are topologically equivalent to $\mathcal{T}_{1/3}$.*

Theorem 8.12 in this paper is related to 3.7: a certain class of "nonexistent components" (cf. 5.1) can only be missing in the tree at internal angle 1/2.

§4 Internal addresses

In order to reach a certain parameter (e.g., the landing point of a given external ray) from the main epicycloid, one must walk through a certain sequence of hyperbolic sectors. By listing the periods and sector numbers of some of them we obtain a sequence called the internal address. These are introduced and studied in [L-S-1] using kneading sequences of the external angles.

4.1 Kneading sequences of parameter classes. We first extend the definition of kneading sequence from angles to parameters in the Mandelbrot set (or actually, to parameter classes in $\mathcal{M}^d/\sim$; cf. 3.1). Every parameter in $\partial\mathcal{M}^d$ is an accumulation point of some external rays, so each parameter class $c \in \mathcal{M}^d/\sim$ has some external angles. If they are *nonperiodic*, the angles all have the same kneading sequence in $\{\mathbf{0}, \mathbf{1}, \ldots \mathbf{d-1}\}^{\mathbb{N}}$ (by Lemma 2.6.) This sequence is defined as $K(c)$.

Parameters with *periodic* external angles can also be given kneading sequences. By the Structure Theorem 2.8, every hyperbolic component $\mathcal{H}$ has two rays landing at its root and $d-2$ at its co-roots. We name the angles $\theta_1, \ldots, \theta_d$ in the natural order, starting and ending at the root point $\widehat{c}$. These d angles all have the same period k under the d-tupling map σ_d.

Furthermore, $K_d^+(\theta) = K_d^-(\varphi)$ for any two angles θ, φ with equal periods whose parameter rays are not separated by any wake boundary (2.6). This is the case for each pair of rays bounding the wake of a sector $\mathcal{H}^s$ of $\mathcal{H}$. We call the angles θ_s, θ_{s+1} of these rays the *external arguments* of $\mathcal{H}^s$, and

$$K_d^+(\theta_s) = K_d^-(\theta_{s+1}) =: K_d(\mathcal{H}^s) \tag{4.2}$$

its kneading sequence (as in [L-S-1]). The symbol $K_d(\theta_s)_{nk}$ changes at each θ_s, but since there are d of them, it returns after one round; if $K_d(\theta_1)_{nk} = {}^{\mathbf{j+1}}_{\mathbf{j}}$, then $K_d(\theta_d)_{nk} = {}^{\mathbf{j}}_{\mathbf{j-1}}$ ($j \in \mathbb{Z}_d$). Therefore

$$K_d^-(\theta_1) = K_d^+(\theta_d) =: K_d(\widehat{c}) =: \widehat{K}_d(\mathcal{H}), \tag{4.3}$$

and we call this the *root sequence* of the hyperbolic component $\mathcal{H}$. For example, the root sequence of the five-periodic component in $\mathcal{M}^3$ shown in Figure D is $\overline{\mathbf{10210}}$ (whereas the kneading sequences for the two sectors of the component are $\overline{\mathbf{10211}}$ and $\overline{\mathbf{10212}}$).

4.4 Internal address. The *rough internal address* of any parameter $c \in \mathcal{M}^d$ is the sequence of integers

$$n_1 \rightarrowtail n_2 \rightarrowtail \ldots, \tag{4.5}$$

defined as follows: $n_1 = 1$, $c_1 = 0$. Among all pairs of periodic parameter rays separating c from c_j, exactly one pair has minimal period, n_{j+1}, and they land at a parameter c_{j+1}.

Lavaurs' Lemma [Lav] guarantees that c_j is unique at each step, and that the sequence is strictly increasing. The point c_j is obviously the root of some hyperbolic component $\mathcal{H}_j$, and c sits in the wake of one of its sectors, s_j. Numbering the sectors of $\mathcal{H}_j$ from 1 through $d-1$ counter-clockwise from the root, as in 4.1, one gets the *internal address* of c,

$$A(c) := n_1(s_1) \rightarrowtail n_2(s_2) \rightarrowtail \ldots. \tag{4.6}$$

4.7 Remark. All parameters in a hyperbolic sector share a finite internal address, so we can speak of internal addresses of sectors. Skipping the last sector number we get an address of the form $1(s_1) \rightarrowtail \cdots \rightarrowtail n_j(s_j) \rightarrowtail k$, which refers to the whole hyperbolic component and its boundary, except the rootpoint.

Primitive parabolic points $\widehat{c}$ have infinite internal addresses, and the rootpoint's address is also called the *root address* of the hyperbolic component. Bifurcation points (or roots of satellite components), on the other hand, share the finite address of the parent sector. All parameters not in closures of hyperbolic components have infinite addresses.

4.8 Address classes. Several parameter classes in different parts of $\mathcal{M}^d/\sim$ may share an internal address; for example, all the four five-periodic satellites of the main cardioid of $\mathcal{M}^2$ have internal address $1 \rightarrowtail 5$. Such components could be distinguished by the (rational) internal angles p_j/q_j at which the combinatorial arc crosses each sector $H_j^{s_j}$ in the internal address; *angled internal address* determines a component uniquely, as shown in [L-S-1].

But the internal address turns out to determine the denominators q_j. Furthermore, if an internal address is realized by a hyperbolic sector with some sequence of numerators, it is also realized if one replaces any of the numerators p_j with any other number p'_j less than and coprime to q_j. Hence, any internal address is either realized by any possible set of angles or not at all.

To keep the notation simpler, we therefore consider just internal addresses without angles, and identify all parameters sharing one. This yields another equivalence relation, denoted by $\simeq$. Each element of the quotient space $\mathcal{M}^d/\simeq$ consists of one or several parameter classes in $\mathcal{M}^d/\sim$ (as defined in 3.1). (By the words "parameter", "component" etc. we will generally refer to elements of the space $\mathcal{M}^d/\simeq$.)

4.9 LSA. Given two parameter classes $\mathcal{C} \prec \mathcal{A}$, we can use Lemma 2.6 to find the minimal period of hyperbolic components on the combinatorial arc between them, by comparing their kneading sequences; it is the index n at which $\widehat{K}_d(\mathcal{A})$ and $K_d(\mathcal{C})$ first differ. The kneading sequence of the n-periodic sector $\mathcal{B} \in]\mathcal{C}, \mathcal{A}[$ is then obtained by repeating the first n entries of $\widehat{K}_d(\mathcal{A})$.

Comparing iteratively $K_d(\mathcal{B})$ (or $\widehat{K}_d(\mathcal{B})$) to the original kneading sequences, one finds a sequence of hyperbolic sectors with increasing periods and nested wakes approaching $\mathcal{A}$ from below (or $\mathcal{C}$ from above, respectively). We call this procedure the *Lau – Schleicher Algorithm (LSA)* [L-S-1, 12.2]. In particular, any parameter (class) $\mathcal{A}$ is above some sector $\mathcal{C}^s$ of the main epicycloid, so comparing the kneading sequence of $\mathcal{A}$ to $K_d(\mathcal{C}^s) = \overline{\mathbf{s}}$, LSA yields its internal address.

4.10 Example. (See Figures B and D.) Comparing the root sequence $\overline{\mathbf{10210}}$ to the kneading sequence $\mathbf{11111}\ldots$ of the first sector of the main epicycloid, 1(1), one finds the first difference at digit $2 = n_2$: $s_2 = 0 - 1 = 3 - 1 = 2$. The original root sequence then differs from $\mathbf{101010}\ldots$ first at $3 = n_3$, and $s_3 = 2 - 1 = 1$. The first difference from $\mathbf{102102}\ldots$ occurs at $6 > 5$, so the internal address of this hyperbolic component is $1(1) \rightarrowtail 2(2) \rightarrowtail 3(1) \rightarrowtail 5$.

§5 Formal Addresses and Formal Kneading Sequences

The Lau–Schleicher Algorithm actually works more generally: it is not necessary to assume a given sequence to occur as the kneading sequence or internal address of some hyperbolic sector. Therefore we can study the Mandelbrot set as a subspace of a more general symbol space.

5.1 Definitions.

- A *d-kneading sequence* is a sequence $\mathbf{a} = a_1 a_2 a_3 \cdots \in \{\mathbf{0}, \mathbf{1}, \ldots, \mathbf{d-1}\}^{\mathbb{N}}$ with $a_1 \neq \mathbf{0}$. We denote by Σ^d the space of all such sequences equipped with the metric [Ate]

$$|\mathbf{a} - \mathbf{b}| := \sum_{i=1}^{\infty} \frac{|a_i - b_i|}{d^i}. \tag{5.1a}$$

- A *rough address* is an increasing sequence of integers starting with 1.
- A *(formal) d-address* is a sequence $A = n_1(s_1) \rightarrowtail n_2(s_2) \rightarrowtail \dots$, where $n_1 \rightarrowtail n_2 \rightarrowtail \dots$ is a rough address and $s_j \in \mathbb{Z}_d \setminus \{0\}$ for all j. Denote the address space by Λ^d.

The metric (5.1a) will give a topological structure for the address space. Many formal addresses appear as the internal addresses of hyperbolic sectors or other parameter classes, but not all. Here, these nonrealizable addresses are thought to be inhabited by *nonexistent* parameters. Hence we may identify, e.g., a hyperbolic sector with its address, and study later whether it exists or not.

5.2 Addresses to kneading sequences and back. When used for turning a kneading sequence into an address, the Lau–Schleicher Algorithm can be written explicitly as follows:

Given a kneading sequence $\mathbf{a} := a_1a_2a_3\dots \in \Sigma^d$, *set* $n_1 = 1$. *For all* $i \in \mathbb{N}$, n_i *is the first index at which the two sequences* $\mathbf{a}$ *and* $\mathbf{b} = \overline{a_1 \dots a_{n_{(i-1)}}}$ *differ; set* $s_i := b_{n_i} - a_{n_i}$. *This yields a formal address* $A = 1(s_1) \rightarrowtail n_2(s_2) \rightarrowtail \dots \in \Lambda^d$.

The procedure is obviously reversible and independent of whether the given sequences actually occur in parameter classes of $\mathcal{M}^d$. Hence LSA defines a bijective mapping $\varkappa_d$ from the set Λ^d of formal addresses to the space Σ^d of kneading sequences:

$$A = 1(s_1) \rightarrowtail n_2(s_2) \rightarrowtail \dots \in \Lambda^d \implies \varkappa_d(A) := \mathbf{a} = a_1a_2\dots \in \Sigma^d \quad \text{such that}$$

$$\begin{array}{ll} a_1\dots a_{n_2-1} = \mathbf{s}_1\dots\mathbf{s}_1, & a_{n_2} = \mathbf{s}_1 + \mathbf{s}_2; \\ a_1\dots a_{n_j-1} = (\overline{a_1\dots a_{n_{(j-1)}}})|_{n_j-1}, & a_{n_j} = (\overline{a_1\dots a_{n_{(j-1)}}})_{n_j} + \mathbf{s}_j \quad \forall j. \end{array} \tag{5.3}$$

The bijection $\varkappa_d : \Lambda^d \to \Sigma^d$ induces a combinatorial and topological structure to the set Λ^d from Σ^d. The space thus obtained contains the factor space $\mathcal{M}^d/\simeq$ (defined in 4.8) as a proper subset.

5.4 Finite addresses. *Finite* formal d-addresses $A = 1(s_1) \rightarrowtail \dots \rightarrowtail n(s)$ ending with $n \in \mathbb{N}$ form a subspace $\Lambda^d_n \subset \Lambda^d$; $n =: |A|$ is called the *length* of the address A. They get mapped by $\varkappa_d$ to *periodic* kneading sequences with exact period n; the set $\dot{\Sigma}^d_n := \varkappa_d(\Lambda^d_n)$ is a proper subset of $\Sigma^d_n := \{\mathbf{a} : a_i = a_{i+n}\ \forall i \in \mathbb{N}\}$.

For each initial word $a_1a_2\dots a_{n-1}$ ($a_j \in \{\mathbf{0}, \mathbf{1}, \dots, \mathbf{d-1}\}$), exactly one choice for the next entry, a_n, does *not* stop LSA yielding the number n to the address, so the remaining $d-1$ choices each do give an n-periodic sequence corresponding to an element of $\dot{\Sigma}^d_n$. The number of formal d-addresses of length n is thus

$$\#\dot{\Sigma}^d_n = \#\Lambda^d_n = d^{n-2}(d-1)^2$$

($d-1$ choices for $j=1$ and $j=n$, d choices for $1<j<n$). Not making the last choice means just referring to all the $d-1$ sectors of one hyperbolic component (whether this component exists or not). Hence we may call formal addresses of the type $A = 1(s_1) \rightarrowtail \dots \rightarrowtail n_i(s_i) \rightarrowtail n$ *component addresses* (in accordance with 4.8). The corresponding kneading sequence is denoted by replacing the last symbol in the periodic word by an asterisk: $\mathbf{a} = \overline{a_1a_2\dots a_{n-1}*}$.

The number of component addresses is thus $d^{n-2}(d-1)$. This number can be compared to the number of existing hyperbolic components with period n, which is $(d^n-2)/d$ if n is prime and less otherwise: all the d^n-2 external rays with angle period n, or strictly dividing n, land at components with the same period in groups of d by the Structure Theorems [Mil-2] [Ebe]. Taking into account that some addresses are realized by several components (cf. 4.8), one can estimate the number of nonexistent components in Λ_n^d for each d, n.

5.5 Example. The periodic words of elements of Σ_6^2 and the corresponding formal addresses are listed below. The left column contains the $2^{6-2}\cdot 1 = 16$ elements of $\dot{\Sigma}_6^2$. The first number shows how many six-periodic hyperbolic components have each one as its internal address. Their total number is $[(2^6-2)-(2^3-2)-(2^2-2)]/2 = 27$.

2	**111110**	$1 \rightarrowtail 6$	**111111**	1
4	**111100**	$1 \rightarrowtail 5 \rightarrowtail 6$	**111101**	$1 \rightarrowtail 5 \rightarrowtail 10 \rightarrowtail 11 \rightarrowtail \dots$
2	**111010**	$1 \rightarrowtail 4 \rightarrowtail 6$	**111011**	$1 \rightarrowtail 4 \rightarrowtail 8 \rightarrowtail 10 \rightarrowtail \dots$
2	**111000**	$1 \rightarrowtail 4 \rightarrowtail 5 \rightarrowtail 6$	**111001**	$1 \rightarrowtail 4 \rightarrowtail 5 \rightarrowtail 9 \rightarrowtail 10 \rightarrowtail 11 \rightarrowtail \dots$
2	**110111**	$1 \rightarrowtail 3 \rightarrowtail 6$	**110110**	$1 \rightarrowtail 3$
2	**110100**	$1 \rightarrowtail 3 \rightarrowtail 5 \rightarrowtail 6$	**110101**	$1 \rightarrowtail 3 \rightarrowtail 5 \rightarrowtail 8 \rightarrowtail 9 \rightarrowtail 11 \rightarrowtail \dots$
2	**110010**	$1 \rightarrowtail 3 \rightarrowtail 4 \rightarrowtail 6$	**110011**	$1 \rightarrowtail 3 \rightarrowtail 4 \rightarrowtail 7 \rightarrowtail 9 \rightarrowtail 10 \rightarrowtail \dots$
2	**110000**	$1 \rightarrowtail 3 \rightarrowtail 4 \rightarrowtail 5 \rightarrowtail 6$	**110001**	$1 \rightarrowtail 3 \rightarrowtail 4 \rightarrowtail 5 \rightarrowtail 8 \rightarrowtail 9 \rightarrowtail 10 \rightarrowtail \dots$
2	**101011**	$1 \rightarrowtail 2 \rightarrowtail 6$	**101010**	$1 \rightarrowtail 2$
2	**101000**	$1 \rightarrowtail 2 \rightarrowtail 5 \rightarrowtail 6$	**101001**	$1 \rightarrowtail 2 \rightarrowtail 5 \rightarrowtail 7 \rightarrowtail 8 \rightarrowtail 11 \rightarrowtail \dots$
1	**101111**	$1 \rightarrowtail 2 \rightarrowtail 4 \rightarrowtail 6$	**101110**	$1 \rightarrowtail 2 \rightarrowtail 4 \rightarrowtail 8 \rightarrowtail 10 \rightarrowtail \dots$
0	**101100**	$1 \rightarrowtail 2 \rightarrowtail 4 \rightarrowtail 5 \rightarrowtail 6$	**101101**	$1 \rightarrowtail 2 \rightarrowtail 4 \rightarrowtail 5 \rightarrowtail 7 \rightarrowtail 8 \rightarrowtail 10 \rightarrowtail \dots$
1	**100101**	$1 \rightarrowtail 2 \rightarrowtail 3 \rightarrowtail 6$	**100100**	$1 \rightarrowtail 2 \rightarrowtail 3$
1	**100110**	$1 \rightarrowtail 2 \rightarrowtail 3 \rightarrowtail 5 \rightarrowtail 6$	**100111**	$1 \rightarrowtail 2 \rightarrowtail 3 \rightarrowtail 5 \rightarrowtail 7 \rightarrowtail 8 \rightarrowtail 9 \rightarrowtail \dots$
1	**100011**	$1 \rightarrowtail 2 \rightarrowtail 3 \rightarrowtail 4 \rightarrowtail 6$	**100010**	$1 \rightarrowtail 2 \rightarrowtail 3 \rightarrowtail 4 \rightarrowtail 7 \rightarrowtail 8 \rightarrowtail 9 \rightarrowtail \dots$
1	**100000**	$1 \rightarrowtail 2 \rightarrowtail 3 \rightarrowtail 4 \rightarrowtail 5 \rightarrowtail 6$	**100001**	$1 \rightarrowtail 2 \rightarrowtail 3 \rightarrowtail 4 \rightarrowtail 5 \rightarrowtail 7 \rightarrowtail 8 \rightarrowtail \dots$

5.6 Remark. For all d and n, $\dot{\Sigma}_n^d$ is a proper subset of $\dot{\Sigma}_n^{d+1}$. Mapping a kneading sequence with $\varkappa_{d+1}$ naturally yields the same rough address as $\varkappa_d$, but the sector numbers may be different. For example, the kneading sequence $\mathbf{c} = \overline{\mathbf{100110}} \in \dot{\Sigma}_6^2$ gets mapped as follows:

$$\begin{aligned}
\varkappa_2^{-1}(\mathbf{c}) &= 1(1) \rightarrowtail 2(1) \rightarrowtail 3(1) \rightarrowtail 5(1) \rightarrowtail 6(1) \\
\varkappa_3^{-1}(\mathbf{c}) &= 1(1) \rightarrowtail 2(2) \rightarrowtail 3(2) \rightarrowtail 5(1) \rightarrowtail 6(2) \\
\varkappa_4^{-1}(\mathbf{c}) &= 1(1) \rightarrowtail 2(3) \rightarrowtail 3(3) \rightarrowtail 5(1) \rightarrowtail 6(3) \\
\varkappa_5^{-1}(\mathbf{c}) &= 1(1) \rightarrowtail 2(4) \rightarrowtail 3(4) \rightarrowtail 5(1) \rightarrowtail 6(4) \\
\varkappa_6^{-1}(\mathbf{c}) &= 1(1) \rightarrowtail 2(5) \rightarrowtail 3(5) \rightarrowtail 5(1) \rightarrowtail 6(5)
\end{aligned}$$

5.7 Root sequence. Let $B \in \Lambda_n^d$ be an address and $\varkappa_d(B) = \mathbf{b} = \overline{b_1 \dots b_n} \in \dot{\Sigma}_n^d$ its kneading sequence. We define its *root sequence* $\widehat{\varkappa_d}(B) := \widehat{\mathbf{b}} := \overline{b_1 \dots b_n'}$ by replacing b_n with the one symbol b_n' such that $\widehat{\mathbf{b}} \notin \dot{\Sigma}_n^d$.

The root sequence $\widehat{\mathbf{b}}$ has the following alternatives. If it has period exactly n, then $\varkappa_d^{-1}(\widehat{\mathbf{b}})$ must be an infinite address skipping n; the set of such sequences is denoted by $\Sigma_{n,\infty}^d$. Unless n is a prime number, the period of $\widehat{\mathbf{b}}$ may be a proper divisor k of n, and the corresponding address may again be either infinite or finite with length k. Thus we have a disjoint union

$$\Sigma_n^d = \dot{\Sigma}_n^d \cup \Sigma_{n,\infty}^d \cup \bigcup_{k|n, k<n} \Sigma_k^d.$$

If the address B is realized by an existing hyperbolic sector $\mathcal{B}$, then $\widehat{\mathbf{b}}$ is its root sequence as defined above (4.3), and $\varkappa_d^{-1}(\widehat{\mathbf{b}})$ is its root address (4.7).

5.8 Primitive and satellite addresses. The internal address and kneading sequence of a *satellite* hyperbolic component $\mathcal{B}$ with period $n = qk$ are of the form

$$B = 1(s_1) \rightarrowtail \cdots \rightarrowtail k(s) \rightarrowtail qk,$$
$$K_d(\mathcal{B}) = \varkappa_d(B) = \overline{(c_1 \dots c_{k-1} c_k)^{q-1} c_1 \dots c_{k-1} *}$$

where $\widehat{K}_d(\mathcal{B}) = \overline{c_1 \dots c_k}$ is the kneading sequence of the k-periodic parent component $\mathcal{C}$ where $\mathcal{B}$ bifurcates from. On the other hand, the internal address and kneading sequence of a *primitive* hyperbolic component are not of this form: instead, its root sequence has the same exact period $\langle \mathcal{B} \rangle$ and translates in LSA into an infinite sequence of sectors approaching it from below.

We now extend this concept for formal addresses and thus for nonexistent components. Thus, an address $B \in \Lambda^d_n$ (and the component $\mathcal{B}$) is

- *primitive* if its root sequence $\widehat{\mathbf{b}} \in \Sigma^d_{n,\infty}$, and
- *satellite* if $\widehat{\mathbf{b}} \in \dot{\Sigma}^d_k$ for some $k \mid n$.

5.9 Shadow satellites. For the root sequence of a formal address B, there is a third possibility: $\widehat{\mathbf{b}} \in \Sigma^d_{k,\infty}$ for some $k \mid n$. Then $\mathbf{b} \in \dot{\Sigma}^d_{qk}$ consists of a primitive root address $\overline{c_1 \dots c'_k} \in \Sigma^d_{k,\infty}$ chopped just before the index $n = qk$ and c_{qk} changed. Such a component $\mathcal{B}$ is a *shadow satellite* of the k-periodic sector $\mathcal{C}$ with kneading sequence $\overline{c_1 \dots c_k} \in \dot{\Sigma}^d_k$ ($\mathcal{C}$ may or may not exist).

Of the sequences in Σ^d_6 (compare to example 5.5),

$$\varkappa_d{}^{-1}(\overline{\mathbf{10110j}}) = 1(1) \rightarrowtail 2(d-1) \rightarrowtail 4(1) \rightarrowtail 5(d-1) \rightarrowtail 6(j-1) \quad (j \neq 1)$$

is an example of a shadow satellite's sector. The root sequence of its parent is $\varkappa_d{}^{-1}(\overline{\mathbf{101}}) = 1(1) \rightarrowtail 2(d-1) \rightarrowtail 4(1) \rightarrowtail 5(d-1) \rightarrowtail 7(1) \rightarrowtail 8(d-1) \rightarrowtail 10(1) \rightarrowtail 11(d-1) \dots$.

5.10 Lemma. *Shadow satellites are nonexistent.*

Proof. Suppose we had a component $\mathcal{B}$ with period a proper multiple qk of k such that $\widehat{\varkappa}_d(B) = \overline{b_1 b_2 \dots b_k} \in \Sigma^d_k \setminus \dot{\Sigma}^d_k$.

All hyperbolic components are either primitive or satellite. If $\mathcal{B}$ were a primitive component, the exact period of $\widehat{\varkappa}_d(B)$ would be qk; if it were a satellite of a k-periodic sector, then $\widehat{\varkappa}_d(B)$ would be the kneading sequence of this sector and thus would have to belong to $\dot{\Sigma}^d_k$. $\square$

5.11 Lemma. *For any natural numbers k and $q \geq 2$, every primitive hyperbolic component (existent or not) with period k has a shadow satellite of period qk. Conversely, the parent component of any shadow satellite is primitive.*

Proof. Let the parent component have kneading sequence $\mathbf{c} := \overline{c_1 c_2 \dots c_{k-1} *}$ and address $C = 1(s_1) \rightarrowtail \cdots \rightarrowtail k$. Then all sequences $\overline{c_1 c_2 \dots c_k}$ with $c_k \in \mathbb{Z}_d$ except one, say $\overline{c_1 c_2 \dots c'_k}$, correspond to the $d-1$ sectors of the same component (and thus have exact period k). If C has a shadow satellite, its kneading sequence is $\overline{(c_1 c_2 \dots c'_k)^{q-1} c_1 c_2 \cdots *}$ for some $q \in \mathbb{N}$.

If C is primitive, then the root sequence $\overline{c_1 c_2 \dots c'_k}$ also has exact period k. Now $\overline{c_1 c_2 \dots c'_k} \in \Sigma^d_k \setminus \dot{\Sigma}^d_k$, so $\overline{(c_1 c_2 \dots c'_k)^{q-1} c_1 c_2 \dots c_k} \in \dot{\Sigma}^d_{qk}$ corresponds to a shadow satellite address.

If C is a satellite address, then the exact period of the root sequence, $\overline{c_1c_2\dots c_k'} = \overline{(c_1c_2\dots c_n)^r} = \overline{c_1c_2\dots c_n}$, is a proper divisor $n = k/r$ of k. This formal kneading sequence belongs to the n-periodic parent sector $\mathcal{N}$ of $\mathcal{C}$, and the sequence $\overline{(c_1c_2\dots c_k')^{q-1}c_1c_2\dots c_k} = \overline{(c_1c_2\dots c_n)^{qr-1}c_1c_2\dots c_n'}$ belongs to a qrn-periodic satellite of $\mathcal{N}$, rather than to a shadow satellite. Therefore satellite components do not have shadow satellites. □

5.12 Remark. We have thus found one infinite class of nonexistent hyperbolic components, namely the shadow satellites of primitive components. It follows that every primitive component has one set of satellites bifurcating from each of its $d-1$ sectors, and one more set of shadow satellites.

The following somewhat technical result describes a regularity property of primitive root addresses: the sequence of differences between consecutive numbers in the rough address, $(n_{i+1}-n_i)_{i\in\mathbb{N}}$, is (pre)periodic.

5.13 Lemma. *If $\mathbf{b} = \overline{b_1\dots b_n} \in \Sigma^d_{n,\infty}$ $(n \geq 3)$ and the corresponding root address is $\varkappa^{-1}(\mathbf{b}) = 1(s_1) \rightarrowtail n_2(s_2) \rightarrowtail \dots$, then a number $N > 2n$ occurs in the address if and only if $N = n_i + tn$ for some $i, t \in \mathbb{N}$ such that $n < n_i < 2n$. Moreover, the sector number $s_N = s_i$.*

Proof. We make two preliminary observations. First, none of the n_i can be an integer multiple of n; otherwise the address would be finite. Second, there must be some i such that $1 < n_i < n$; otherwise $\mathbf{b} = \overline{b_1} \in \dot{\Sigma}^d_1$ contrary to the assumption. Denoting the indices so that $1 < n_j < n < n_{j+1} \leq n_h < 2n < n_{h+1}$, the statement will follow from these two results:

(i) $n_h = n_j + n$.
(ii) If $n_m = n_i + n$ for some i, m, then $n_{m+1}(s_{m+1}) = (n_{i+1}+n)(s_{i+1})$.

To prove (i), note that by definition (5.3) the first $n_{j+1}-1 \geq n$ figures in $\mathbf{b}$ and $\overline{b_1\dots b_{n_j}}$ agree; in particular,

$$b_1\dots b_{n-n_j} = b_{n_j+1}\dots b_n = b_{n_j+n+1}\dots b_{2n}$$

(the last equality follows from the n-periodicity of $\mathbf{b}$). Now the first $2n$ figures in $\mathbf{b}$ and $\overline{b_1\dots b_{n_j+n}}$ agree, so the maximal period number less than $2n$ in the root address, n_h, is at most n_j+n. On the other hand, $n-n_j$ is the maximal number of same figures in the beginning and end of the word $b_1\dots b_n$, and thus also the word $b_1\dots b_n\dots b_{2n}$, so $n_h = 2n-(n-n_j) = n_j+n$. In the following expression, the braced subwords are equal:

$$\mathbf{b} = \underbrace{b_1b_2\dots}b_{n_{(j+1)}-n_j}\dots b_{n_j}\underbrace{\dots b_n\dots}b_{n_{(j+1)}}\dots b_{n_h}\underbrace{\dots b_{2n}\dots}b_{n_{(h+1)}}\dots$$

Proof of (ii): If $n_{i+1} = n_i+1$, then $b_1 + s_{i+1} = b_{n_{(i+1)}} = b_{n_{(i+1)}+n}$. The first index at which $\mathbf{b}$ and $\overline{b_1\dots b_{n_i+n}}$ differ is n_{m+1}; the $(n_i+n+1)^{\text{th}}$ figure of the latter is $b_1 = b_{n_{(i+1)}+n} - s_{i+1}$, so $n_{m+1} = n_{i+1}+n$. Otherwise $n_{i+1}-n_i > 1$, and

$$b_{n_i+1}\dots b_{n_{(i+1)}-1} = b_1\dots b_{n_{(i+1)}-n_i-1},\ b_{n_{(i+1)}} - s_{i+1} = \overline{b_1\dots b_{n_i}}|_{n_{(i+1)}} = b_{n_{(i+1)}-n_i}.$$

Adding n to the indices preserves the equality, so the first $n_{i+1}-n_i-1$ figures of $\mathbf{b}$ equal $b_{n_m+1}\dots b_{n_{(i+1)}+n-1}$, and $b_{n_{(i+1)}-n_i} = b_{n_{i+1}+n} - s_{i+1}$. Hence the first difference between $\mathbf{b}$ and $\overline{b_1 \dots b_{n_m}}$ occurs at index $n_{m+1} = n_m + n_{i+1} - n_i = n_{i+1} + n$; the difference $b_{n_{(m+1)}} - \overline{b_1 \dots b_{n_m}}|_{n_{(m+1)}} = s_{i+1}$ is the sector number s_{m+1}. $\square$

5.14 Remark. In other words, starting from some index less than $2n$, the same period numbers modulo n and the same sector numbers modulo d keep recurring infinitely. Example 5.5 shows that the sequence $\{n_1, \dots, n_j\}$ is not necessarily equal to $\{n_{j+1}, \dots, n_h\}$ modulo n, so the increment sequence $(n_{i+1} - n_i)_{i \in \mathbb{N}}$ may be strictly preperiodic. For the twelve infinite root addresses on the right column of 5.5, the increment sequences are

$$\begin{array}{llll} 4, \overline{5, 1}; & 2, \overline{2, 3, 1}; & \overline{1, 3, 2}; & \overline{1, 1, 2, 2} \\ 3, \overline{4, 2}; & \overline{2, 1, 3}; & 1, \overline{2, 4}; & \overline{1, 1, 1, 3} \\ 3, \overline{1, 4, 1}; & 2, \overline{1, 1, 3, 1}; & \overline{1, 2}; & \overline{1, 1, 1, 1, 2}. \end{array}$$

Note that the sum of the increments in the periodic part (overlined above) is always the exact period of the kneading sequence; six, in this example, except for the shadow satellite.

5.15 Period and subaddress. We will extend some more definitions regarding existent parameters and their internal addresses to formal addresses – and thus nonexistent parameter classes – like we did above with primitiveness and satelliteness.

The *period* of a nonexistent component is the length of its address; $\langle \mathcal{B} \rangle := |B|$.

Any formal address $B = 1(s_1) \rightarrowtail n_2(s_2) \rightarrowtail \dots \in \Lambda^d$ determines a (finite or infinite) sequence of finite *subaddresses* $B_j = 1(s_1) \rightarrowtail n_2(s_2) \rightarrowtail \dots \rightarrowtail n_j(s_j) \in \Lambda^d_{n_j}$ of B.

5.16 Partial ordering of addresses. We define the "above"-relation recursively as follows:

- If B is a subaddress of A, then $B \prec A$.
- If A, C are formal addresses and $C \prec A$, then comparing the kneading sequences $\widehat{\varkappa}_d(A)$ and $\varkappa_d(C)$ LSA yields a unique formal kneading sequence and the corresponding finite address B; now $C \prec B \prec A$.
- If B is an infinite address and $B_j \prec A$ for all the subaddresses B_j of B, then $B \prec A$.

These conditions are evidently satisfied by the internal addresses of existent parameter classes that are ordered by the natural partial relation (3.1), so this new relation is its extension. Combinatorial arc $[C, A]$ and visibility are now defined similarly as in 3.2.

Any formal address can be joined to $1(s_1)$ by a unique combinatorial arc. Note that uniqueness follows directly from the definition (second point of 5.16): given two symbolic sequences, the place of their first difference determines a unique address between them (cf. 5.2). In the Mandelbrot set we needed the Lavaurs' lemma (3.3) to guarantee that the corresponding *hyperbolic sector* is also unique; nonexistent sectors, on the other hand, are just formal addresses, for which there is no ambiguity to begin with. Hence, Λ^d is a partially ordered, simply connected topological space.

5.17 Remark. If A and C are any formal addresses, one can use LSA to study whether they are comparable in $\prec$. If their root kneading sequences differ, then necessarily after a finite number of iterations one either finds that $A \in \overline{[1(s_1), C]}$ or else there is a finite address $B \in]1(s_1), C[$ such that $B \notin]1(s_1), A[$.

For example, consider the two addresses $A = 1 \rightarrowtail 2 \rightarrowtail 4 \rightarrowtail 5 \in \Lambda^2_5$ and $B = 1 \rightarrowtail 2 \rightarrowtail 4 \rightarrowtail 6 \rightarrowtail 9 \in \Lambda^2_9$.
Clearly B cannot be above A, because the minimal period of components between B and $C_4 := 1 \rightarrowtail 2 \rightarrowtail 4$ is six and not five. Thus either $B \prec A$, or they diverge at C_4 or above it. Now the root sequence of A and its root address are, respectively, $\overline{\mathbf{10111}}$ and $1 \rightarrowtail 2 \rightarrowtail 4 \rightarrowtail 6 \rightarrowtail 7 \rightarrowtail 9 \ldots.$
Because B does not contain number seven, B or the branching point must be found between C_6 and C_7. Thus we compare

$$\widehat{\varkappa}_2(C_7) = \mathbf{1011111\ 1011111}\ldots \quad \text{with} \quad \varkappa_2(C_6) = \mathbf{101111\ 101111}\ldots$$

to find their first difference at the 8^{th} digit. Because B does not contain number eight either, we then compare

$$\widehat{\varkappa}_2(C_8) = \mathbf{10111110\ 10111110}\ldots \quad \text{with} \quad \varkappa_2(C_6) = \mathbf{101111\ 101111}\ldots$$

They first differ at the tenth digit; $C_4 \prec C_6 \prec C_{10} \prec C_8 \prec C_7 \prec A$ and $C_4 \prec C_6 \prec B$. If B were below A, number 9 should have occurred by now. Since it has not, B must branch off the arc $]1, A[$ at some point.

The visible-tree algorithm we present in §6 will enable us find such combinatorial relations much more easily, making this kind of complicated calculations largely unnecessary.

5.18 Remark. The root sequences of two different components can only be identical if $\mathcal{A}$ is a shadow satellite of $\mathcal{C}$: if we have a n-periodic root sequence $a_1a_2\ldots$, we get the kneading sequence of the component by replacing either a_n or a_{2n}, a_{3n}, etc. by '$*$'. The word $a_1...a_{n-1}*$ repeated periodically yields a primitive component, whereas $a_1...a_na_1...a_{qn-1}*$ gives its shadow satellite of period qn. Now neither component is above the other, but $\mathcal{A}$ is visible from (and hence above) any subaddress of $\mathcal{C}$'s root whose length is more than $\langle\mathcal{A}\rangle$.

5.19 Vanishing point. Every formal address starts with $1(s_1)$ for some s_1, which is realized by a sector of the main epicycloid. If an address A *is* realized, then so must be anything *below* it; that is, the whole closure of the combinatorial arc $[1(s_1), A]$.

On the other hand, if an address $E \in \Lambda^d$ is *not* realized, then neither is any address *above* it (by uniqueness of combinatorial arcs). Therefore there must be a unique point in the closure of $[1(s_1), E]$ which divides the combinatorial arc into realizable and nonrealizable parts. This parameter $v \in \mathcal{M}^d/\simeq$, at which the arc jumps from the complex plane into nonexistence, is called the *vanishing point* of the nonexistent component $\mathcal{E}$.

5.20 Shadow component. We have seen that a component $\mathcal{E}$ is nonexistent if it is a shadow satellite or sits above one. We call such components *shadow components* and find more of them in the next section. The definitions above imply the following characterization:

5.21 Lemma. *Let $\mathcal{E}$ be a nonexistent component. Then $\mathcal{E}$ is a shadow component if and only if its vanishing point is the root of some existent component $\mathcal{B}$.* $\square$

5.22 Remark. Being a shadow component is thus a sufficient condition for a component to be nonexistent for all degrees $d \geq 2$. In the quadratic case $d = 2$, this is also a necesssary condition by [B-S]. For higher degrees this is open.

Given a kneading sequence **a**, Henk Bruin and Dierk Schleicher define [B-S, 2.2] a mapping $\rho : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$, $\quad \rho(n) := \inf\{k > n : a_k \neq a_{k-n}\}$ and $orb_\rho(n) := n \rightarrowtail \rho(n) \rightarrowtail \rho \circ \rho(n) \rightarrowtail \ldots$; $orb_\rho(1)$ is then the corresponding formal address as defined here in 5.2. A kneading sequence **e** and its address $E \in \Lambda^2_n$ are said to *fail the admissibility condition for period* n if [B-S, 2.2]

(1) E does not contain n
(2) If $k < n$ divides n, then $\rho(k) \leq n$
(3) If $r \leq n$ is congruent to $\rho(n)$ modulo n, then $orb_\rho(k)$ contains n.

This is shown to be equivalent to the nonexistence of the component E. These conditions also imply that E is a shadow component [B-S, 6.5].

§6 Growing Formal Trees

We know from the theory of internal addresses [L-S-1] that a kneading sequence contains information of which way one had to walk in order to reach the sector with that kneading sequence. It turns out that the kneading sequence also contains information about what lies ahead. There is a Visible-tree algorithm (VTA, 6.4) by which one can find the combinatorial structure of the tree of all formal addresses that are "potentially" visible from a given base sector – existent or not.

Unfortunately, some components given by VTA do not actually exist in $\mathcal{M}^d$ even if the base does. However, we now grow the whole trees with our algorithm and study their properties for their own right, then "prune" nonexistent twigs in §8.

6.1 Visibility over a sector. Let $\mathcal{A}$ and $\mathcal{B}$ be hyperbolic sectors with periods m and n respectively, such that $\mathcal{B} \prec \mathcal{A}$, $n > m$, and no sector on $]\mathcal{B}, \mathcal{A}[$ has period less than m. Then we say that $\mathcal{A}$ is *visible over* $\mathcal{B}$. If no sector on $]\mathcal{B}, \mathcal{A}[$ has period less than n, then $\mathcal{A}$ is *immediately visible over* $\mathcal{B}$.

These definitions are motivated by the fact that such a sector $\mathcal{A}$ is visible from both $\mathcal{B}$ and any sector $\mathcal{C}$ below $\mathcal{B}$ from where $\mathcal{B}$ is, in turn, visible (by definitions 3.2 and 5.16; note that we need not assume the sectors in question to exist). Moreover, no sector on the combinatorial arc $]\mathcal{B}, \mathcal{A}[$ is visible from $\mathcal{C}$ if $\mathcal{A}$ is immediately visible from $\mathcal{B}$.

In any visible tree, each hyperbolic sector is obviously immediately visible over the preceding sector. Our tree-growing algorithm will be based on the following lemma, which gives a necessary condition for a hyperbolic sector to sit above another, immediately visible over it: the periodic word of the lower one begins and ends with the same subword.

6.2 Lemma. *If $\mathcal{A}$ is a hyperbolic sector immediately visible over another sector $\mathcal{B}$ with $\langle\mathcal{A}\rangle = m < n = \langle\mathcal{B}\rangle$, then the periodic word $K(\mathcal{B})|_n = b_1 \ldots b_n$ has the same subword of length $l := n - m$ in both ends, i.e., $b_1 \ldots b_l = b_{m+1} \ldots b_n$. Moreover, $m \nmid n$.*

Proof. By definition of immediate visibility, $\mathcal{A}$ and $\mathcal{B}$ are not separated by parameter rays with periods less than or equal to n, so (by 2.6) the first n digits of the sequences $\widehat{K}(\mathcal{A})$ and $K(\mathcal{B})$ must agree. Hence the kneading sequences of $\mathcal{A}$ and $\mathcal{B}$ differ exactly at indices that are multiples of m:

$$b_1 b_2 \dots b_m \dots b_n = a_1 a_2 \dots a'_m a_{m+1} \dots a_n. \tag{6.2a}$$

If $n = jm$ for some $j \in \mathbb{N}$, this would imply $K(\mathcal{B}) = \overline{(a_1 a_2 \dots a'_m)^j} = \overline{a_1 a_2 \dots a'_m} \in \Sigma^d_{m,\infty}$, which is impossible. Because $\widehat{K}(\mathcal{A})$ is m-periodic, (6.2a) yields

(6.2b)

$$b_{m+1} \dots b_{2m} \dots b_n = a_{m+1} \dots a'_{2m} \dots a_n = a_1 \dots a'_m \dots a_{n-m} = b_1 \dots b_l. \quad \square$$

It follows directly that an address *not* satisfying this subword condition can have no components with less periods above it, so it must be realized by a *narrow* hyperbolic sector (if at all). (The converse statement, that all narrow sectors have narrow addresses, will be proved in 8.4.) We can now define:

6.3 Narrow address. A formal address $B \in \Lambda^d_n$ is *narrow* if its periodic word $b_1 \dots b_n$ does not begin and end with the same subword of any length.

Lemma 6.2 can be used iteratively to find the addresses of all hyperbolic components that *may* be visible from a given sector $\mathcal{C}$, starting with the satellite components, proceeding to primitive components immediately visible over the satellites, and so on.

Finding all formal addresses that satisfy the necessary subword-condition, we can now construct the *formal tree* based at a given finite formal address, from the stem to branch-tips. Because we are only dealing with the internal addresses and not with any analytical information about the parameter plane or even external angles, this works for nonexistent base sectors as well as existent ones.

6.4 Visible Tree Algorithm (VTA). Let $\mathcal{C}$ at the formal d-address $C = 1(s_1) \rightarrowtail \dots \rightarrowtail k(s_k)$ with $\varkappa_d(C) = \overline{c_1 c_2 \dots c_k}$ be the base sector. Fix denominator q of the internal angle. For each sector number s, find the tree's stem (the q-satellite)

$$C^s_q = 1(s_1) \rightarrowtail \dots \rightarrowtail k(s_k) \rightarrowtail qk(s)$$

with $\varkappa_d(C^s_q) = \overline{(c_1 c_2 \dots c_k)^{q-1} c_1 c_2 \dots (c_k + s)}$ by changing the qk^{th} digit in the kneading sequence from $c_{qk} = c_k$ into $c_k + s$.

Find all numbers l_o such that $c_1 \dots c_{l_o} = c_{qk-l_o+1} \dots (c_{qk}+s)$; there are obviously a finite number of such l_o's. For each l_o there is a primitive address A immediately above the stem with period $m := qk - l_o$ and root kneading sequence $\overline{c_1 \dots c_m} =: \overline{a_1 \dots a'_m}$. Replacing $c_m = a'_m$ with the other numbers $a_m \in \mathbb{Z}_d$, one gets the kneading sequences of respective sectors of $\mathcal{A}$.

Then, for all addresses found in this way, find every number l_1 such that $a_1 \dots a_{l_1} = a_{m-l_1+1} \dots a_m$, chop off the last l_1 digits in the periodic word and change the $(m - l_1)^{\text{th}}$ one to obtain the kneading sequences $\overline{a_1 \dots (a_{m-l_1} + s)}$ $(s \in \mathbb{Z}_d \setminus \{0\})$, and so on, until no more such l's can be found.

We call the resulting set the *formal tree* $\widetilde{\mathcal{T}}_{\cdot/q}$ of C (or $\mathcal{C}$). It is not necessarily combinatorially equivalent to the "real" tree $\mathcal{T}_{p/q}$, but it contains a subtree which is. The following examples together with Figure G illustrate the action of the algorithm.

6.5 Example. Consider the address $C = 1(1) \rightarrowtail 3(1) \rightarrowtail 6(1) \rightarrowtail 7(1) \in \Lambda^3_7$ with $\varkappa_3(C) = \overline{\mathbf{1121102}}$. The periodic word **1121102 1121100** of the first sector of the satellite $C^1_2 \in \Lambda^3_{14}$ has no common subwords at both ends, so the sector is narrow. The second sector has "**1**" at both ends, so there is a 13-periodic component above it.

Its first sector has periodic word **1121102 112111**, which starts and ends with both "**11**" and "**1**", giving rise to adjacent addresses of lengths $13 - 2 = 11$ and $13 - 1 = 12$, respectively. These are both narrow.

The second sector of the address in Λ^3_{13} has periodic word **1121102 112112** with "**112**" at both ends, so it precedes a ten-periodic component whose first sector (**1121102 110**) is narrow and the second precedes a pair of adjacent narrow components with periods eight and nine. Using the joker symbol $*$ as in 5.4, we write briefly:

1121102 1121100	$1(1) \rightarrowtail 3(1) \rightarrowtail 6(1) \rightarrowtail 7(1) \rightarrowtail 14(1)$
1121102 1121101	$1(1) \rightarrowtail 3(1) \rightarrowtail 6(1) \rightarrowtail 7(1) \rightarrowtail 14(2)$
1121102 112111	$1(1) \rightarrowtail 3(1) \rightarrowtail 6(1) \rightarrowtail 7(1) \rightarrowtail 13(1)$
1121102 112$*$	$1(1) \rightarrowtail 3(1) \rightarrowtail 6(1) \rightarrowtail 7(1) \rightarrowtail 11$
1121102 1121$*$	$1(1) \rightarrowtail 3(1) \rightarrowtail 6(1) \rightarrowtail 7(1) \rightarrowtail 12$
1121102 112112	$1(1) \rightarrowtail 3(1) \rightarrowtail 6(1) \rightarrowtail 7(1) \rightarrowtail 13(2)$
1121102 110	$1(1) \rightarrowtail 3(1) \rightarrowtail 6(1) \rightarrowtail 7(1) \rightarrowtail 10(1)$
1121102 111	$1(1) \rightarrowtail 3(1) \rightarrowtail 6(1) \rightarrowtail 7(1) \rightarrowtail 10(2)$
1121102 $*$	$1(1) \rightarrowtail 3(1) \rightarrowtail 6(1) \rightarrowtail 7(1) \rightarrowtail 8$
1121102 1$*$	$1(1) \rightarrowtail 3(1) \rightarrowtail 6(1) \rightarrowtail 7(1) \rightarrowtail 9$

6.6 Lemma. *The formal tree $\widetilde{\mathcal{T}}_{\cdot/2}$ of the address*

$$C = 1(s_1) \rightarrowtail n_2(s_2) \rightarrowtail \cdots \rightarrowtail n_{k-1}(s_{k-1}) \rightarrowtail n_k(s_k)$$

contains a chain of formal addresses A_j such that $A_j \in \Lambda^d_{n_k+n_j}$ for all $j \in \{1, \ldots, k\}$ and $A_1 \succ A_2 \succ \cdots \succ A_{k-1} \succ A_k = C_2$. Each A_j is immediately visible over A_{j+1}, and the sector number of A_j equals $d - s_j$.

(This lemma was inspired by an observation by Dierk Schleicher, p.c.)

Proof. In the periodic word $c_1 \ldots c_{n_2} \ldots c_{n_{(k-1)}} \ldots c_{n_k}$ of C, by definition,

$$c_{n_k} - s_k = (\overline{c_1 \ldots c_{n_{(k-1)}}})_{n_k} = (\overline{c_1 \ldots c_{n_{(k-1)}}})_{n_k - n_{(k-1)}} = c_{n_k - n_{k-1}}$$

(c_i modulo d, as always). On the other hand, the periodic word of the satellite A_k's sector number r_k ends with $c_{n_k} + r_k$. Thus one sector, namely number $d - s_k = r_k$, has the subword $c_1 \ldots c_{n_k - n_{(k-1)}}$ at both ends of its periodic word. This means that there is a formal address A_{k-1} with length $2n_k - (n_k - n_{k-1}) = n_k + n_{k-1}$ immediately visible over the stem.

Continuing backwards inductively, if the tree contains a component address A_j of length $n_k + n_j$, then its sector number r_j has periodic word

$$c_1 c_2 \ldots c_{n_k} c_{n_k+1} \ldots c_{n_k + n_{(j-1)}} c_{n_k + n_{(j-1)}+1} \ldots (c_{n_k+n_j} + r_j).$$

The last digit equals $c_{n_j} + r_j = c_{n_j - n_{(j-1)}} + s_j + r_j$, so the subword $c_1 \ldots c_{n_j - n_{j-1}}$ begins and ends the periodic word of sector number $r_j = d - s_j$ of A_j, giving rise to a component address A_{j-1} of length $n_k + n_{j-1}$ immediately above A_j. $\square$

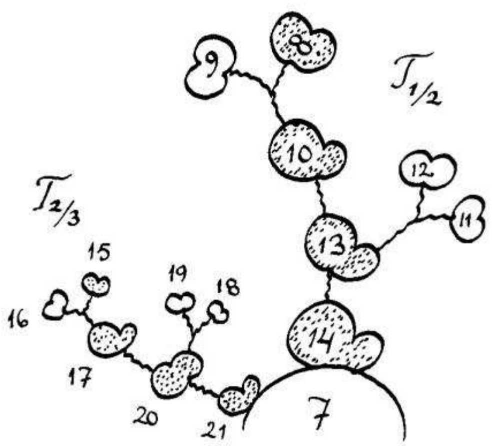

FIGURE G

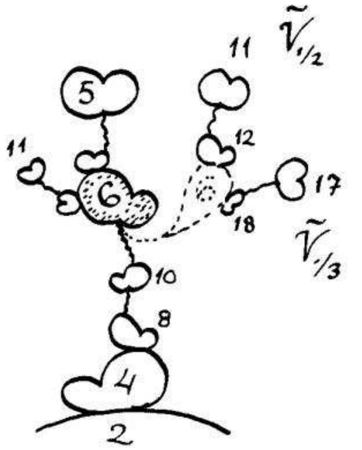

FIGURE H

6.7 Secondary trunk. The chain in Lemma 6.6 of n_k sectors on the combinatorial arc from sector C to its continuation $C \rightarrowtail n_k + 1$ is called the *secondary trunk* of the tree $\widetilde{\mathcal{T}}_{\cdot/2}(C)$.

For narrow sectors, the primary (3.4) and secondary trunks are obviously the same set (e.g., Figure G), and it can be called just the tree's *trunk*. For non-narrow sectors, the secondary trunk may be a subset of the primary trunk (like the tree $\widetilde{\mathcal{T}}^2_{\cdot/2}$ in Figure J) or the two trunks may branch off each other (like $\widetilde{\mathcal{T}}^1_{\cdot/2}$ in Figure J).

As we saw, for any $s \in \mathbb{Z}_d \setminus \{0\}$, the $(d-s)^{\text{th}}$ sector of the 2-satellite of s^{th} sector of the base component has periodic word $c_1c_2 \ldots c_{n_k-1}(c'_{n_k} + s)c_1c_2 \ldots c_{n_k-1}c'_{n_k}$, where $\overline{c_1c_2 \ldots c'_{n_k}} = \widehat{\varkappa}(C) \in \Sigma^d_{n_k,\infty}$. Because these $d-1$ words of length $2n_k$ differ only at index n_k, the VT-algorithm proceeds identically for all sectors s as long as the addresses remain longer than the base. We have thus proved

6.8 Corollary. *Let component $\mathcal{C}$ (existent or not) be the base. If a sector $\mathcal{A}$ with period $n > n_k = \langle \mathcal{C} \rangle$ appears above a sector $\mathcal{B}$ in the secondary trunk of one base sector $\mathcal{C}^s$, then another sector $\mathcal{A}'$ of same period n appears above the corresponding sector $\mathcal{B}'$ with period $\langle \mathcal{B} \rangle$ in the secondary trunk of any other base sector $\mathcal{C}^r$.* □

For narrow sectors $\mathcal{C}^s, \mathcal{C}^r$ of the same base component this means that the subtrees stemming at the 2-satellite's sectors $(d-s), (d-r)$ respectively, are combinatorially equivalent.

See how the tree in the previous example (6.5) has a chain of addresses with lengths $7+7, 7+6, 7+3, 7+1$ and sector numbers $3 - s_j$. These are shaded in Figure G (here $k = 4$, $n_k = 7$, $j \in \{1,2,3,4\}$). See also Figure J, where the secondary trunks and the addresses of lengths $13, 17 (> 10)$ branching off it are similar for the two sectors of the 10-periodic base component, but the addresses with lengths less than ten (five and six on the first sector, nine on the second) branch off the tree trunks at different places.

The next lemma restricts the difference between the periods of two consecutive components in the tree – or, on another point of view, restricts the periods of sectors from where such a pair of consecutive sectors can be visible:

6.9 Lemma. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be hyperbolic sectors with periods m, n, k respectively and $l := n - m$. If $\mathcal{B}$ is visible from $\mathcal{C}$ and $\mathcal{A}$ is immediately visible over $\mathcal{B}$, then $l < k$.*

Proof. The sectors $\mathcal{A}$ and $\mathcal{B}$ are in the same formal tree $\widetilde{\mathcal{T}}_{\cdot/q}$ of $\mathcal{C}$ for some q; $m < n \leq qk$. By 6.4, $\widehat{K}(\mathcal{B})$ and $K(\mathcal{C})$ agree up to at least the n^{th} digit. Hence $a_n = b_n \neq c_n$ and

$$c_1 c_2 \dots (c_n + s) = b_1 b_2 \dots b_n \tag{6.9a}$$

where $s \neq 0$ is the sector number of $\mathcal{B}$. First, assume that $\boxed{l > k}$, which forces $n = m + l > k$. Combining (6.2a), (6.2b), and (6.9a), we obtain

(6.9b)

$$c_1 \dots c_k \dots c_l = c_{m+1} \dots c_{m+k} \dots (c_n + s) \implies c_{k+1} \dots c_l = c_{m+k+1} \dots (c_n + s).$$

On the other hand, $c_j = c_{k+j}\ \forall j$, so

$$c_{k+1} \dots c_l = c_1 \dots c_{l-k} = c_{m+1} \dots c_{m+l-k}.$$

But since $m + l = n$, the last digit is $c_{n-k} = c_n \neq c_n + s$, contradicting (6.9b).

Finally, assume $\boxed{l = k}$. Then $m = ik + r$ and $n = (i+1)k + r$ for some integers r and i such that $0 \leq i \leq q-2$, $1 \leq r \leq k-1$ ($r > 0$ because $\mathcal{A}$ cannot be a satellite of $\mathcal{C}$ or $\mathcal{B}$). Now (6.2a) and (6.9a) imply

$$\begin{aligned} b_1 \dots b_{ik} \dots b_{(i+1)k} \dots b_n &= (c_1 \dots c_k)^i c_1 \dots c_k c_1 \dots (c_r + s) \\ = a_1 \dots a_{ik} \dots a'_m \dots a_n &= (c_1 \dots c_k)^i c_1 \dots c_r c_1 \dots (c_k + t) \end{aligned}$$

where $t \neq 0$ is the sector number of $\mathcal{A}$. But by (6.2b), the last digit should equal $b_n = b_l = b_k = c_k$ (because $k = l < n$ and the kneading sequences of $\mathcal{C}$ and $\mathcal{B}$ agree up to the n^{th} digit). $\square$

6.10 Shadow trees. VT-Algorithm can equally well be used for finding all formal addresses with lengths at most qk which are above the *shadow satellites* of the given base address C (realizable or not), instead of the proper satellites.

Using the same notation as above, let $c'_k \neq c_k$ such that $\overline{c_1 \dots c_{k-1} c'_k} \in \Sigma^d_{k,\infty}$. Fixing integer $q \geq 2$, find all sectors $\widehat{C}^s_q$ of the shadow q-satellite; their kneading sequences are $\varkappa_d(\widehat{C}^s_q) = \overline{(c_1 \dots c'_k)^{q-1} c_1 \dots (c'_k + s)}$ ($s \in \mathbb{Z}_d \setminus \{0\}$).

Then find all numbers l such that $c_1 \dots c_l = c_{qk-l+1} \dots (c'_k + s)$. For each such l there is an address $B \in \Lambda^d_{qk-l}$ with $\widehat{\varkappa}_d(B) = \overline{(c_1 \dots c'_k)^{q-1} c_1 \dots c_{k-l}}$ immediately above the stem; replace the last digit by other numbers in $\mathbb{Z}_d$ to get the kneading sequences of all sectors of B. Find common parts in both ends of their periodic words, and so forth.

The combinatorial space thus obtained is called the *shadow tree* $\widetilde{\mathcal{V}}_{\cdot/q}$ of C. Because they are above shadow satellites which are nonexistent, the whole shadow trees are nonexistent.

6.11 Example. Each shadow q-tree of the (realizable) formal address $A := 1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 6$ with periodic word **12101**$*$ consists of the shadow q-satellite and one primitive address of length $(q-1)\cdot 6+5$ above the second sector of the stem; see Figure H.

121012 121010	$1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 8(2) \rightarrowtail 10(1) \rightarrowtail 12(1)$
121012 121011	$1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 8(2) \rightarrowtail 10(1) \rightarrowtail 12(2)$
121012 1210$*$	$1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 8(2) \rightarrowtail 10(1) \rightarrowtail 11$
121012 121012 121010	$1(1) \rightarrowtail \cdots \rightarrowtail 10(1) \rightarrowtail 14(2) \rightarrowtail 16(1) \rightarrowtail 18(1)$
121012 121012 121011	$1(1) \rightarrowtail \cdots \rightarrowtail 10(1) \rightarrowtail 14(2) \rightarrowtail 16(1) \rightarrowtail 18(2)$
121012 121012 1210$*$	$1(1) \rightarrowtail \cdots \rightarrowtail 10(1) \rightarrowtail 14(2) \rightarrowtail 16(1) \rightarrowtail 17$

6.12 Remark. In Remark 5.18 we noticed that shadow satellites are visible from the base's root subaddresses with sufficiently large periods. Now we see that the same is true for all primitive components in each shadow tree; they are, by definition, visible *over* a shadow satellite of the base sector $\mathcal{C}$. Any address $E \in \widetilde{\mathcal{V}}_q(\mathcal{C})$ is of the form

$$1(s_1) \rightarrowtail n_2(s_2) \rightarrowtail \cdots \rightarrowtail n_j(s_j) \rightarrowtail h,$$

where n_j is the maximal period of root subaddresses of the base component $\mathcal{C}$ such that $n_j < h = |E|$. (By 5.13, $\langle\mathcal{C}\rangle$ does not appear in the root address of $\mathcal{C}$, and hence not in E either.)

Now if $\mathcal{D}$ is a hyperbolic sector realizing some root subaddress of the base $\mathcal{C}$ such that $\langle\mathcal{D}\rangle \geq n_j$, then the nonexistent component $\mathcal{E}$ is visible from $\mathcal{D}$ and hence appears in the formal tree given by VTA. In particular, if we move our vantage point to longer and longer finite subaddresses, more and more formal addresses in the shadow trees of the base $\mathcal{C}$ become visible.

For example, see Figures A and H. The shadow tree of the component at $1(1) \rightarrowtail 2(1) \rightarrowtail 4(2) \rightarrowtail 6$ contains components with periods $9, 11, 12$. From the subaddress $A = 1(1) \rightarrowtail 2(1) \rightarrowtail 4(2)$, these three are invisible because the satellite sector $A \rightarrowtail 8(2)$ is blocking the view. Moving there, the 9-periodic component becomes visible but the ones with periods $11, 12$ are still behind the primitive sector $A \rightarrowtail 8(2) \rightarrowtail 10(2)$. Seen from *there*, the two finally appear, along with a 17-periodic component which eventually will turn out to belong to another shadow tree.

6.13 Lemma. *If every sector C^s of C is narrow, then for each s, all but one sector of the 2-satellite of C^s are narrow too. Moreover, all shadow satellites are narrow, so the shadow trees of C contain no primitive addresses.* $\square$

(This follows directly from definitions.) The exceptional non-narrow sector of the 2-satellite is number $d-s$, by Lemma 6.6.

At this point we can prove a result, "Formal Translation Principle", which generalizes the equivalence statement 3.6 for formal trees and also for shadow trees. (Note that this is not true in general for real visible trees.)

6.14 Theorem. *Let C be a formal address. For any integers q, q' greater than 1, the formal trees $\widetilde{\mathcal{T}}_{\cdot/q}$ and $\widetilde{\mathcal{T}}_{\cdot/q'}$ based at C are combinatorially equivalent. The shadow trees $\widetilde{\mathcal{V}}_{\cdot/q}$ and $\widetilde{\mathcal{V}}_{\cdot/q'}$ of C are also combinatorially equivalent.*

We divide the proof into a few lemmas about the combinatorial structure of various formal trees and shadow trees.

6.15 Lemma. *If there is a formal address A of length $m < k$ above an address $C \in \Lambda^d_k$, then A is above the 2-satellite of C.*

Proof. We may assume that m is maximal in the sense that there are no other addresses with periods less than k between C and A. Then A belongs to some formal visible tree $\widetilde{T}_{./q}$ of C. (Any other addresses with periods less than k above C must then sit above one of such "lowest visible" addresses.) We will show that $q = 2$.

The periodic word $c_1 \dots c_k$ of the base C begins and ends with some subword of length $k - m$. Comparing the root sequence of A, $\overline{c_1 \dots c_m}$, to $\overline{c_1 \dots c_k}$, following the LSA, one finds the chain of addresses in the visible tree connecting A to C. The process will end at the stem, whose period is some multiple of k. We must show that it ends at $2k$.

After the first step, the root kneading sequences to be compared with $\overline{c_1 \dots c_k}$ all have periods greater than k. Therefore the process is equivalent to finding the root address of the base C. By the same argument and notation as in Lemma 5.13, we see that after finitely many steps the first difference will occur at some index $k + n_i$ where $n_i < k$ appears in the address $C := 1(s_1) \rightarrowtail \cdots \rightarrowtail n_j(s_j) \rightarrowtail k(s)$ $(i \le j)$. In particular, the algorithm will stop at the period $k + n_j$.

The next step will be different from that in 5.13, because now the $2k^{\text{th}}$ digit is $c_k \ne c_{k-n_j}$. Therefore the two sequences $\overline{c_1 \dots c_k}$ and $\overline{c_1 \dots c_{k+n_j}}$ differ at index $2k$, so the algorithm ends. $\square$

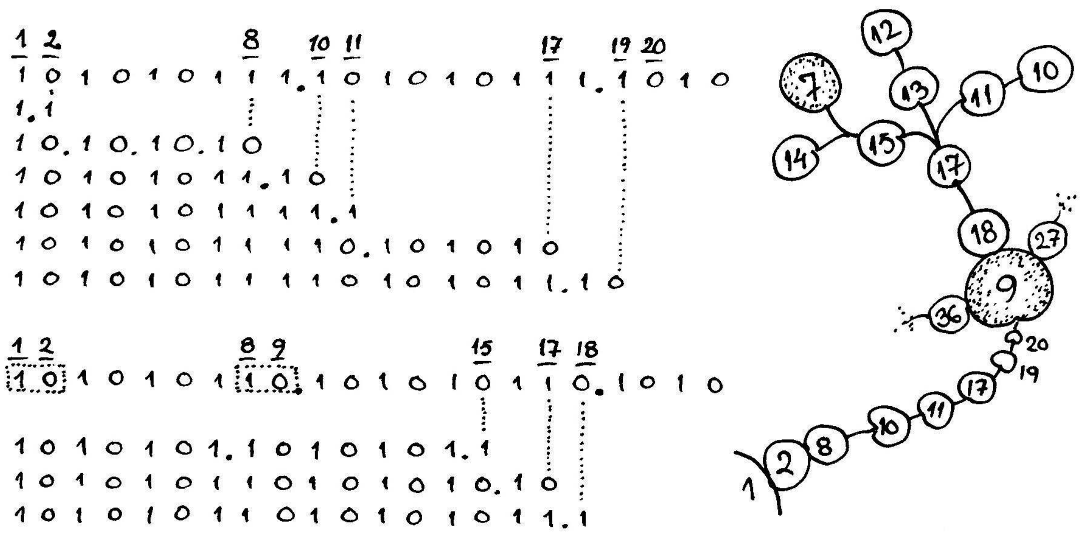

Figure I

Note that the argument is also similar to the one in the proof of Lemma 6.6, only the algorithm runs in opposite direction. Figure I shows the Lau–Schleicher Algorithm from the main cardioid up to a component at $C := 1 \rightarrowtail 2 \rightarrowtail 8 \rightarrowtail 9 \in \Lambda^2_9$, and from $A := 1 \rightarrowtail 2 \rightarrowtail 7$ back down to C. See the secondary trunk of the tree with the ten-periodic component at the top, and the primary trunk with A at the top.

The next lemma provides a kind of induction step on the partial order of components, for the proof of Theorem 6.14.

6.16 Lemma. *Let $C \in \Lambda_k^d$ be a base address. The following are equivalent:*

- *The formal tree $\widetilde{\mathcal{T}}_{\cdot/q}$ of C contains the addresses $B \in \Lambda_n^d$ and $A \in \Lambda_m^d$ immediately visible over B.*
- *The formal tree $\widetilde{\mathcal{T}}_{\cdot/q+1}$ of C contains the addresses $B' \in \Lambda_{k+n}^d$ and $A' \in \Lambda_{k+m}^d$ immediately visible over B', with the sector numbers of B and B' equal.*

Proof. (Clearly, $m < n \le qk$.)

(i) Assume that there are addresses $B \in \widetilde{\mathcal{T}}_{\cdot/q}$ of length $n \le qk$ and $B' \in \widetilde{\mathcal{T}}_{\cdot/q+1}$ of length $k+n$, and that the subtrees constructed up to B and B' are equivalent. This is true at least if B and B' are the s^{th} sectors of the two trees' stems, respectively.

Because B and B' are visible from C, the first n figures in the root kneading sequence $\widehat{\varkappa}_d(B)$ must agree with those of $\varkappa_d(C) = \overline{c_1 c_2 \dots c_k}$, and the first $k+n$ figures in $\widehat{\varkappa}_d(B')$ also agree with those of $\varkappa_d(C)$. Denoting the periodic words of B and B' respectively by $b_1 \dots b_n$ and $\beta_1 \dots \beta_{k+n}$, this implies that $b_j = \beta_j = \beta_{k+j}$ $\forall j < n$. Moreover, since the sector numbers are equal, also $b_n = \beta_{k+n} (\neq \beta_n)$.

(ii) If there is a formal address A of length m immediately visible over B, then by Lemmas 6.2 and 6.9, $0 < n - m = l < k$ such that $c_1 \dots c_l$ starts and ends the periodic word of B. Together with (i) this implies that $c_1 \dots c_l$ also starts and ends the periodic word of B', so there is a formal address A' of length $k+n-l = k+m$ above B'. This argument is obviously symmetrical with respect to B and B', so the converse statement is also true. $\square$

6.17 Corollary. *For every $q \ge 2$, the formal q-tree $\widetilde{\mathcal{T}}_{\cdot/q}$ of the address $C = 1(s_1) \rightarrowtail \cdots \rightarrowtail k(s_k) \in \Lambda_k^d$ consists of exactly k component addresses, with lengths in $\{(q-2)k+3, \dots, qk\}$.*

Proof. It follows from 6.16 that the lengths of component addresses in a same tree are all different modulo k, so there are at most k of them in each $\widetilde{\mathcal{T}}_{\cdot/q}$.

If $B \in \widetilde{\mathcal{T}}_{\cdot/2}$, then $3 \le |B| \le 2k$. For each $j = 3, \dots, k$, the tree $\widetilde{\mathcal{T}}_{\cdot/2}$ may contain either an address of length j or one of length $k+j$. We must rule out the possibility that it contains neither.

Every formal address with C as its last subaddress is visible from C, so let $B := 1(s_1) \rightarrowtail \cdots \rightarrowtail k(s_k) \rightarrowtail k+j$ sit in some formal tree $\widetilde{\mathcal{T}}_{\cdot/q}$. If $q \ge 3$, Lemma 6.16 implies that $\widetilde{\mathcal{T}}_{\cdot/q-1}$ contains an address of length $j < k$, so by Lemma 6.15, $q = 3$. Hence B has exactly two possibilities: either 2-tree or 3-tree. In both cases the tree $\widetilde{\mathcal{T}}_{\cdot/2}$ contains exactly k component addresses, and by 6.16, so do all other trees. $\square$

6.18 Remark. It is easy to see that the arguments in the proofs of lemmas 6.16, 6.17 work equally well if "formal tree $\widetilde{\mathcal{T}}_{\cdot/q}$", "satellite C_q^s", and "c_k" are replaced by "shadow tree $\widetilde{\mathcal{V}}_{\cdot/q}$", "shadow satellite $\widehat{C}_q$", and "c_k'", respectively. Hence we have proved Theorem 6.14.

Now we show a couple of results relating the trees of the various sectors of the same base *component*. The following lemma reveals a kind of symmetry between the formal trees and the shadow trees of any formal address, so it will also give us a handy tool for sieving nonexistent components from the formal trees.

6.19 Lemma. *Let $C = 1(s_1) \rightarrowtail \cdots \rightarrowtail n_i(s_i) \rightarrowtail \cdots \rightarrowtail k \in \Lambda_k^d$ be a formal component address and $s \in \{1, \ldots, d-1\}$ a sector number.*
(a) *The following conditions are equivalent:*

- *The formal tree $\widetilde{\mathcal{T}}^s_{./2}$ based at sector C^s contains a formal address B with length $|B| =: n < k$ immediately visible over C.*
- *The formal tree $\widetilde{\mathcal{T}}^r_{./2}$ based at any other sector C^r $(r \neq s)$ contains a B' with $|B'| = k + n$ such that $n \neq n_i$ for all i; B' is immediately visible over the $(s-r)^{\text{th}}$ sector of the stem satellite.*
- *If C is primitive, then the shadow tree $\widetilde{\mathcal{V}}_{./2}$ of C contains a B' with $|B'| = k+n$ $(n \neq n_i\ \forall i)$ immediately visible over the s^{th} sector of the stem satellite. If C is a q-satellite of another address D, then the formal $2q$-tree of D contains a B' with $|B'| = k+n$ $(n \neq n_i\ \forall i)$ immediately visible over the s^{th} sector of the stem satellite.*

(b) *Assuming there are formal addresses B and B' as in (a), the tree containing B has an address $A \succ B$ with length $m < n$ if and only if the tree (formal or shadow) containing B' also has an $A' \succ B'$ with $|A'| = k + m$.*

Proof. The first condition in (a) implies that the number n cannot appear in the address C: otherwise there would be a subaddress $E \prec C$ with length n such that there were no addresses shorter than n between E and C, and not between C and B either, by definition 6.1. Then the last subaddresses of E and B would be equal, so $E = B$. This contradicts definition 5.16.

Denoting the root sequence by $\widehat{\varkappa}_d(C) = \overline{c_1 c_2 \ldots c_{k-1} c'_k}$, each sector s has kneading sequence $\varkappa_d(C^s) = \overline{c_1 c_2 \ldots c_{k-1}(c'_k + s)}$. Then for all $j, r \in \mathbb{Z}_d \setminus \{0\}$, the j^{th} sector of the 2-satellite of sector C^r has periodic word

$$c_1 c_2 \ldots c_{k-n} \ldots c_{k-1}(c'_k + r) c_1 \ldots c_n \ldots c_{k-1}(c'_k + r + j).$$

For $r = 0$, this word belongs to the j^{th} sector of the shadow 2-satellite of C in case C is primitive; otherwise C is a q-satellite of D, and the word belongs to the j^{th} sector of the $2q$-satellite of D (cf. 5.9).

All conditions in (a) are satisfied if and only if $j = s - r$ and $c_1 c_2 \ldots c_{k-n} = c_{n+1} \ldots c_{k-1}(c'_k + s)$ (by 6.2 and 6.10). (Note that this lemma says nothing about B' if it sits in the subtree containing the secondary trunk, because in that case by Lemma 6.6 the satellite sector number would be $d - r$, so $s = d = 0$.)

The assumption in (b) means that $\varkappa_d(B) = \overline{b_1 \ldots b_n} = \overline{c_1 \ldots c'_n} \in \Lambda_n$ and $\varkappa_d(B') = \overline{\beta_1 \ldots \beta_{k+n}} = \overline{c_1 \ldots c'_k c_1 \ldots c'_n} \in \Lambda_{k+n}$. The first condition in (b) is then satisfied if and only if $b_1 \ldots b_{n-m} = b_{m+1} \ldots b_n$; the second, if and only if $\beta_1 \ldots \beta_{(k+n)-(k+m)} = \beta_{k+m+1} \ldots \beta_{k+n}$. Since $\beta_j = c_j = b_j \quad \forall j < n$ and the length of these words is $n - m < n$, the two conditions are equivalent. □

It follows that the trees stemming from different sectors of a same component differ in addresses with lengths *less* than that of the base; see Figure J and recall Lemma 6.8, which said that they all have addresses with the same lengths *greater* than the base.

6.20 Corollary. *Given a base component (address) C and a number $n < |C|$ not appearing in C, there is at most one tree based on C containing a formal address of length n.* □

Define $\widetilde{\mathcal{S}}$ as the tree consisting of the base component and components visible over it (an example is right of Figure J).

6.21 Corollary. *For every $q \geq 2$, the shadow tree $\widetilde{\mathcal{V}}_{\cdot/q}$ of C is combinatorially equivalent to the tree $\widetilde{\mathcal{S}}$ of C.* $\square$

6.22 Example. Let $C = 1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 8(2) \rightarrowtail 10 \in \Lambda^3_{10}$ be the base address with $\varkappa_d(C) = \overline{\mathbf{121012121}*}$. We first grow the formal trees $\widetilde{\mathcal{T}}^s_{\cdot/2}$ for the two sectors of C; see left of Figure J.

1210121210 1210121211	$1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 8(2) \rightarrowtail 10(1) \rightarrowtail 20(1)$
1210121210 12101212$*$	$1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 8(2) \rightarrowtail 10(1) \rightarrowtail 19$
1210121210 1210121212	$1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 8(2) \rightarrowtail 10(1) \rightarrowtail 20(2)$
1210121210 12101210	$1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 8(2) \rightarrowtail 10(1) \rightarrowtail 18(1)$
1210121210 1211	$1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 8(2) \rightarrowtail 10(1) \rightarrowtail 14(1)$
1210121210 12$*$	$1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 8(2) \rightarrowtail 10(1) \rightarrowtail 13$
1210121210 1212	$1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 8(2) \rightarrowtail 10(1) \rightarrowtail 14(2)$
121010	$1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 6(1)$
121011	$1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 6(2)$
1210$*$	$1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 5$
1210121210 10	$1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 8(2) \rightarrowtail 10(1) \rightarrowtail 12(1)$
1210121210 11	$1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 8(2) \rightarrowtail 10(1) \rightarrowtail 12(2)$
1210121210 $*$	$1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 8(2) \rightarrowtail 10(1) \rightarrowtail 11$
1210121210 12101211	$1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 8(2) \rightarrowtail 10(1) \rightarrowtail 18(2)$
1210121210 121012$*$	$1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 8(2) \rightarrowtail 10(1) \rightarrowtail 17$

1210121211 1210121212	$1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 8(2) \rightarrowtail 10(2) \rightarrowtail 20(1)$
...	
1210121211 2	$1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 8(2) \rightarrowtail 10(2) \rightarrowtail 11(1)$
12101212$*$	$1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 8(2) \rightarrowtail 9$
1210121211 1210121210	$1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 8(2) \rightarrowtail 10(2) \rightarrowtail 20(2)$
1210121211 121010	$1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 8(2) \rightarrowtail 10(2) \rightarrowtail 16(1)$
1210121211 121011	$1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 8(2) \rightarrowtail 10(2) \rightarrowtail 16(2)$
1210121211 1210$*$	$1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 8(2) \rightarrowtail 10(2) \rightarrowtail 15$

The first and second tree have components with periods respectively $5, 6; 19$, and $15, 16; 9$, like Lemma 6.19 states. See also how both trees have the trunk 20, 18, 14, 12, 11 in accordance with Lemma 6.6. However, by comparing this example to 6.11 (Figure H) one sees that the components with periods 12 and 11 in the first tree actually belong to a *shadow tree* of the 6-periodic component and are thus nonexistent. (In the second tree, all components do exist; we will see this in §8).

The whole base component $\mathcal{C}$ together with the components with periods less than ten ($5, 6$ above the first sector and 9 above the second) make up the tree $\mathcal{S}$ defined for Corollary 6.21; $\mathcal{S}$ is combinatorially equivalent to the shadow tree $\widetilde{\mathcal{V}}_{\cdot/2}$ of C and, by 6.16, to all other shadow trees as well.

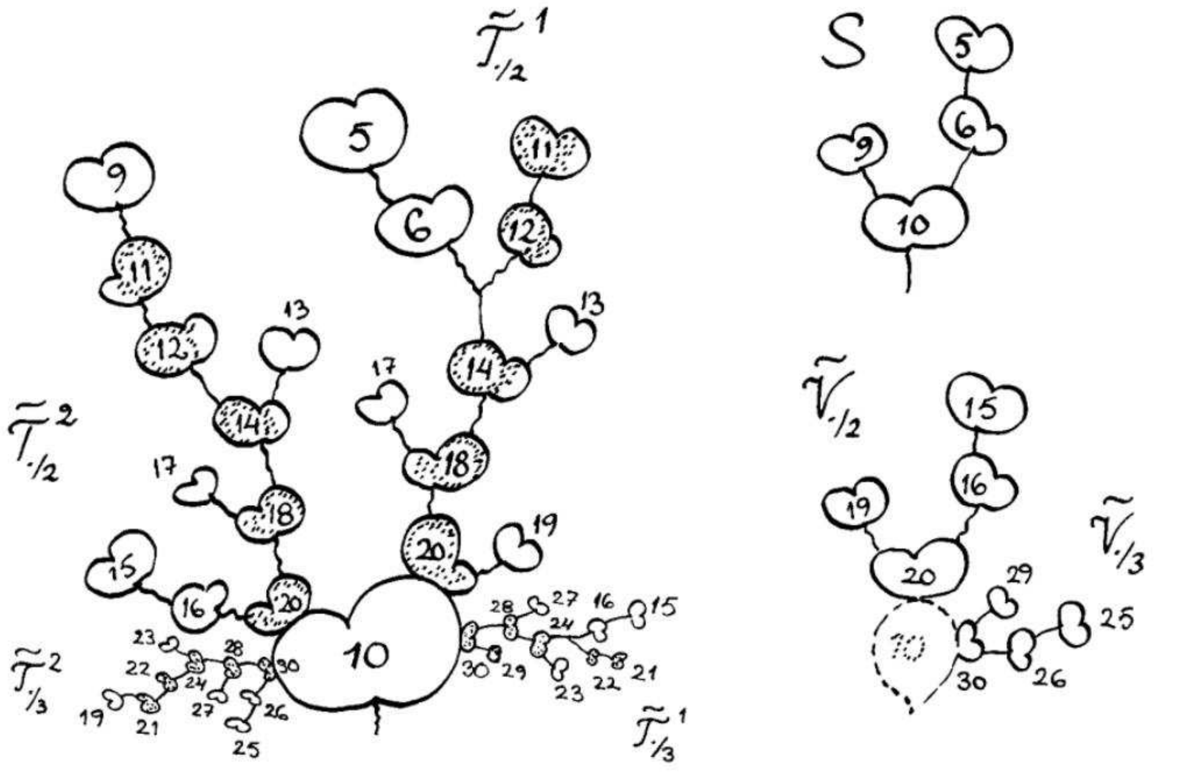

Figure J

1210121212 1210121210	$1(1) \rightarrowtail \cdots \rightarrowtail 8(2) \rightarrowtail 12(2) \rightarrowtail 14(1) \rightarrowtail 18(2) \rightarrowtail 20(1)$
1210121212 121010	$1(1) \rightarrowtail \cdots \rightarrowtail 8(2) \rightarrowtail 12(2) \rightarrowtail 14(1) \rightarrowtail 16(1)$
1210121212 121011	$1(1) \rightarrowtail \cdots \rightarrowtail 8(2) \rightarrowtail 12(2) \rightarrowtail 14(1) \rightarrowtail 16(2)$
1210121212 1210$*$	$1(1) \rightarrowtail \cdots \rightarrowtail 8(2) \rightarrowtail 12(2) \rightarrowtail 14(1) \rightarrowtail 15$
1210121212 1210121211	$1(1) \rightarrowtail \cdots \rightarrowtail 8(2) \rightarrowtail 12(2) \rightarrowtail 14(1) \rightarrowtail 18(2) \rightarrowtail 20(2)$
1210121212 12101212$*$	$1(1) \rightarrowtail \cdots \rightarrowtail 8(2) \rightarrowtail 12(2) \rightarrowtail 14(1) \rightarrowtail 18(2) \rightarrowtail 19$

In the next section we study the widths of wakes of existing hyperbolic components and sectors; this information will tell how much there is room for components in each visible tree, and thus will help to find which components do not exist.

§7 Wake width

The difference of the two external arguments θ_1, θ_2 of a hyperbolic sector $\mathcal{K}$ is called its *width*. Since the angles have the same period k as $\mathcal{K}$, the width must be of the form $|\mathcal{K}| = \theta_2 - \theta_1 = w/(d^k - 1)$ for some integer w.

7.1 Narrow sectors and components. In 3.2, we defined a narrow sector as one whose wake contains no components with less period. This is equivalent to the standard definition: *The sector $\mathcal{K}$ is narrow if $w = 1$.* In order to contain a component with minimal period m, the wake of $\mathcal{K}$ must have width $w/(d^k - 1) > (d-1)/(d^m - 1)$; then $w = 1$ if and only if $m = k + 1$. A hyperbolic component is said to be narrow if all its sectors are.

The following result is fundamental for studying which of the components given by VTA are existent in the Mandelbrot sets, because it tells how much space there is above each hyperbolic sector for other components, but in the general form it has been "folklore". The quadratic case is proved in [Sch-1] using Douady's Tuning Algorithm. Instead of generalizing that method, our proof will be based on a kind of global counting argument. The special case of the main cardioid in $\mathcal{M}^2$ was treated in detail in [Dev] using Farey numbers, like we will. (I was unaware of the work [Dev] until Peter Haïssinsky mentioned it while reading the manuscript of this article.) Another discussion of the main cardioid case, using dyadic expansions of the external angles, is found in [Ate].

7.2 Theorem. *Let $\mathcal{K}$ be a hyperbolic sector of $\mathcal{M}^d$ with period k and width $w/(d^k-1)$. Then each satellite component $\mathcal{C}$ at internal angle p/q has width*

$$\Delta(\tfrac{p}{q}) = \frac{w(d^k-1)}{d^{qk}-1}.$$

7.3 Lemma. *Let $\mathcal{K}$ be a hyperbolic sector with period k and wake-width $w/(d^k-1)$; let the wake contain a hyperbolic component $\mathcal{A}$ with minimal period m. Then*

1. *$w = dr+1$ for some $r \in \mathbb{N}$.*
2. *The number of components with period (dividing) k in the wake of one sector of $\mathcal{A}$ is $r(k,m) = \sum_{1\le i\le (k-1)/m} d^{k-im-1}$.*
3. *If the wake of $\mathcal{K}$ contains no other components with periods less than k, then $w = d(d-1)r(k,m)+1$.*

Proof. (1) Let r be the number of other components with period (dividing) k in the wake of $\mathcal{K}$. By Structure Theorem 2.8, each component has exactly d rays with angle denominator d^k-1 landing at its boundary, and every such ray lands at the boundary of some such component. Thus the wake must contain exactly dr rays with angle denominator d^k-1, so the width has numerator $dr+1$.

(2) The m-periodic component $\mathcal{A}$ must be narrow because m is minimal. The sector-wake therefore has width $1/(d^m-1)$ and contains $dr(k,m)$ rays of the form $t/(d^k-1)$. Now $r(k,m)$ must be the maximal number for which the following holds:

$$\frac{dr(k,m)-1}{d^k-1} < \frac{1}{d^m-1} \iff dr(k,m) < 1 + d^{k-m} + d^{k-2m} + d^{k-3m} + \dots$$

Thus $r(k,m)$ is the integer part of $\sum_{i=1}^{\infty} d^{k-im-1}$; hence $i \le (k-1)/m$.

(3) Since $\mathcal{A}$ has $d-1$ sectors, its wake contains $(d-1)r(k,m)$ components with periods dividing k. The difference between the two ultimate external angles of all these is $[d(d-1)r(k,m)-1]/[d^k-1]$, so $w = d(d-1)r(k,m)+1$. □

7.4 Lemma. *The characteristic interval of any satellite orbit portrait Θ with orbit period k, ray period qk and rotation number p/q has length $\Delta(\frac{p}{q}) = w(\frac{p}{q}) \cdot \frac{d^k-1}{d^{qk}-1}$ with some integer $w(\frac{p}{q})$.*

Proof. Any interval of length Δ is mapped by σ_d^n into an interval of length $d^n\Delta$, possibly covering the image more than once. By 2.3, the complementary intervals $I_1,\dots,I_q$ of A_1 are mapped by σ_d^k so that $|I_{j+p}| = d^k|I_j| \mod 1$ for all $j \in \mathbb{Z}_q$ (thus $\sigma^k(I_j)$ covers I_{j+p} once more than it does other points on the circle). Since $\sum_{j=1}^q |I_j| = 1$, the total length must be a natural number

$$\Delta + d^k\Delta + \dots + d^{(q-1)k}\Delta = \Delta \cdot \frac{d^{qk}-1}{d^k-1}. \qquad \square$$

Now it remains to show that this number $w(p/q)$ depends on neither p nor q, and that the constant equals w.

7.5 Lemma. *If $w(\frac{p}{q}) \neq w(\frac{p'}{q'})$, then $|w(\frac{p}{q}) - w(\frac{p'}{q'})| \geq d$.*

Proof. Lemmas 7.3 and 7.4 imply that $w(\frac{p}{q})(d^k-1) = dr-1$ and $w(\frac{p'}{q'})(d^k-1) = dr'-1$ for some $r, r' \in \mathbb{N}$. The difference $|r-r'| = |w(\frac{p}{q}) - w(\frac{p'}{q'})|(d^k-1)/d$ must also be a natural number, so $|w(\frac{p}{q}) - w(\frac{p'}{q'})|$ is a multiple of d. $\square$

7.6 Farey addition. A *Farey sequence of order n* is the increasing sequence F_n of all (reduced) fractions p/q such that $0 \leq p \leq q \leq n$; for example,

$$F_7 = \frac{0}{1}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{1}{1}$$

[H-W] or [Cha]. If p/q and p'/q' are successive fractions in some F_n, then $\frac{p+p'}{q+q'}$ is called their *mediant*. Fractions in $F_{n+1} \setminus F_n$ are mediants of neighboring fractions in F_n [Cha, Theorem I.8], so all rational numbers on $[0,1]$ are obtained by Farey mediants. Moreover, of three successive fractions in a same Farey sequence, the middle one is the mediant of the other two [Cha, Theorem I.9]. It follows that the mediant of two Farey-adjacent fractions has the minimal denominator of all fractions between them.

7.7 Lemma. *If p/q and p'/q' are successive fractions in F_s such that q is a multiple of q', then $q' = 1$.*

Proof. Now p/q is the mediant of p'/q' and its Farey-neighbor t_1/r_1 of some order less than s, so $q = nq' = q' + r_1$. Iterating this argument, we obtain a sequence (r_i),

$$r_i = (n-i)q' = q' + r_{i+1},$$

of denominators of numbers Farey-adjacent to p'/q' of decreasing order. The sequence stops at $r_{n-1} = q'$. But the only Farey-adjacent pair of numbers with equal denominators are $0/1$ and $1/1$, so $q' = 1$. $\square$

7.8 Lemma. *Let the internal angles p/q and p'/q' of respectively two points c and c' in $\partial\mathcal{K}$ be successive fractions in F_n. Denote by D the domain between the subwakes of $\mathcal{K}$ with roots at c and c', bounded by $\partial\mathcal{K}$ and the two external rays $\mathcal{R}_{\varphi_+(\frac{p}{q})}$ and $\mathcal{R}_{\varphi_-(\frac{p'}{q'})}$ landing at c and c', respectively. The width of D is then*

$$\delta_{q,q'} = \varphi_-(\tfrac{p'}{q'}) - \varphi_+(\tfrac{p}{q}) = \frac{w(\frac{p+p'}{q+q'})(d^k-1)}{(d^{qk}-1)(d^{q'k}-1)}.$$

Proof. (See Figure K.) Because $\delta_{q,q'}$ is the difference of two angles with periods $qk, q'k$, the denominator must be as claimed. If M is the smallest period of hyperbolic components in $D \setminus W(\frac{p+p'}{q+q'})$, then the difference

$$\delta_{q,q'} - \Delta(\tfrac{p+p'}{q+q'}) = \frac{u}{(d^{qk}-1)(d^{q'k}-1)} - \frac{v}{d^{(q+q')k}-1} = \frac{(u-v) + v(d^{-qk} + d^{-q'k}) - \dots}{d^{(q+q')k} - \dots}$$

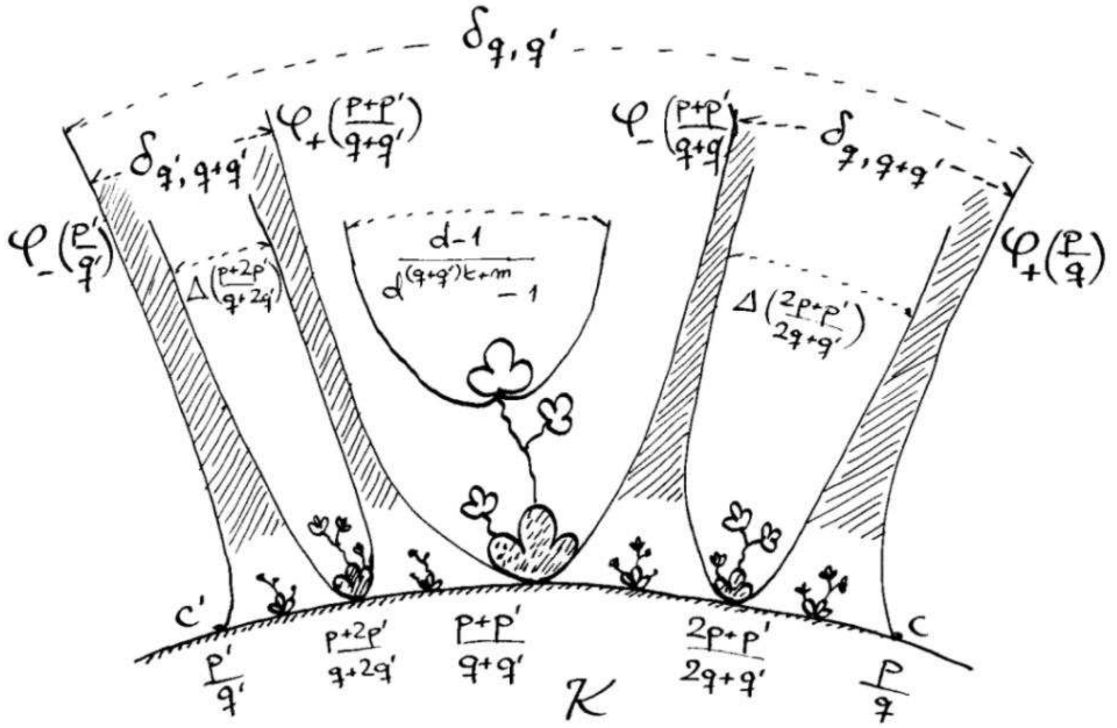

FIGURE K

must be positive and less than $2/(d^{M-1}-1)$, because neither side of the middle wake can contain rays with periods $M-1$ or less. By Lemma 7.4, $v = w(\frac{p+p'}{q+q'})(d^k-1)$. The M-periodic component sits in some visible tree $\mathcal{T}_{\cdot/r}$ of $\mathcal{K}$. By Lemmas 6.16 and 6.17, the *formal tree* stemming at internal angle with denominator r contains primitive addresses of lengths $m' := (r-2)k+m$ where $3 \leq m < 2k$. In particular, if m is the minimal length of addresses visible from $\mathcal{K}$ given by VTA (6.4), then $M \geq m'$. (We need not assume that the components with periods m and m' actually exist.)

The minimal denominator r of all internal angles for satellites in $D \setminus W(\frac{p+p'}{q+q'})$ belongs to the mediant of $\frac{p+p'}{q+q'}$ and either $\frac{p}{q}$ or $\frac{p'}{q'}$, so

$$r = \min\{2q+q', q+2q'\} \geq q+q'+2, \quad M \geq m' = (r-2)k+m \geq (q+q')k+3.$$

We now have upper bound

$$\frac{2}{d^{M-1}-1} < \frac{2}{d^{(q+q')k+2}-1} \leq \frac{1}{d^{(q+q')k+1}-d^{-1}} < \frac{1}{d^{(q+q')k}}$$

for the difference above. This is only possible if $u = v$. $\square$

7.9 Lemma. *For all $q \in \mathbb{N}$, $w(\frac{1}{q}) = w = w(\frac{q-1}{q})$.*

Proof. Angles $1/q$ and $(q-1)/q$ are Farey-adjacent to $0/1$ and $1/1$, respectively, so we may use the previous result 7.8 for $q' = 1$. We prove the lemma for $1/q$; the other case is similar.

The d-tupling map iterated k times fixes the two angles $\theta_\pm$ bounding the sector $\mathcal{K}$, and maps the less characteristic angle of the $\frac{1}{q}$-satellite component to the greater one. Now $\varphi_+^{0/1} = \theta_-$ (and $\varphi_-^{1/1} = \theta_+$), so using 7.4 we have

$$\varphi_-^{1/q} - \delta_{1,q} = \theta_- = \sigma^k(\theta_-) = \sigma^k(\varphi_-^{1/q} - \delta_{1,q}) = \varphi_+^{1/q} - d^k\delta_{1,q}$$

$$\Longrightarrow\ \Delta(\tfrac{1}{q}) = \varphi_+^{1/q} - \varphi_-^{1/q} = (d^k-1)\delta_{1,q} \ \Longrightarrow\ \delta_{1,q} = \frac{w(\frac{1}{q})}{d^{qk}-1}. \tag{7.9a}$$

In particular, for $q=2$, $\delta_{1,2} = w(\frac{1}{2})/(d^{2k}-1)$. Now

$$\theta_+ - \theta_- = \frac{w}{d^k-1} = \Delta(\tfrac{1}{2}) + 2\delta_{1,2} = \frac{w(\frac{1}{2})(d^k-1) + 2w(\frac{1}{2})}{d^{2k}-1} = \frac{w(\frac{1}{2})}{d^k-1},$$

so $\boxed{w(1/2) = w}$. On the other hand, Lemma 7.8 implies

$$\delta_{1,q} = \frac{w(\frac{1+0}{q+1})}{d^{qk}-1}$$

which, together with (7.9a), yields $w(\frac{1}{q+1}) = w(\frac{1}{q})$. The claim now follows by induction with respect to q. □

7.10 Lemma. *For all $q \in \mathbb{N}$ and $p = 2, \ldots, q-2$, $w(\frac{p}{q}) = w$.*

Proof. Every rational number is obtained by the Farey procedure. Assume that

$$\delta_{q,q'} = \frac{w(d^k-1)}{(d^{qk}-1)(d^{q'k}-1)} \quad \text{and} \quad \Delta(\tfrac{p+p'}{q+q'}) = \frac{w(d^k-1)}{d^{(q+q')k}-1}$$

for some pair of successive numbers p/q and p'/q' in F_n. By 7.9, this is true at least when $n=1$.

For the induction step, we calculate the widths $\delta_{q,q+q'}$ and $\delta_{q',q+q'}$ of the domains that should contain the two mediant subwakes next to the one at internal angle $\frac{p+p'}{q+q'}$ (Figure K). By Lemma 7.3, the numerators differ from w by a multiple of d, if at all. Hence for some $i, j \in \mathbb{Z}$,

$$\begin{aligned}\delta_{q,q+q'} + \delta_{q',q+q'} &= \frac{(w+id)(d^k-1)}{(d^{(q+q')k}-1)(d^{qk}-1)} + \frac{(w+jd)(d^k-1)}{(d^{(q+q')k}-1)(d^{q'k}-1)} \\ &= \delta_{q,q'} - \Delta(\tfrac{p+p'}{q+q'}) + \frac{d(d^k-1)}{d^{(q+q')k}-1}\cdot\left(\frac{i}{d^{qk}-1} + \frac{j}{d^{q'k}-1}\right).\end{aligned}$$

The last term must be zero. Unless q is a multiple of q' (or vice versa), this implies $i=j=0$. Thus $w(\frac{2p+p'}{2q+q'}) = w = w(\frac{p+2p'}{q+2q'})$, as claimed.

If q is a multiple of q', then by Lemma 7.7, $q'=1$. All angles Farey-adjacent to $0/1$ or $1/1$ are of the form $1/q$ or $(q-1)/q$, respectively. This case was already proved in 7.9. □

Now $\boxed{w(p/q) = w}$ for all $p/q \in \mathbb{Q}\cap]0,1[$, so we have proved Theorem 7.2. □

The next result is another "weak" or "partial translation principle":

7.11 Corollary. *Every visible tree $\mathcal{T}_{\cdot/q+1}$ of a sector $\mathcal{K}$ contains a subtree which is combinatorially equivalent to $\mathcal{T}_{\cdot/q}$.*

Proof. Denote the wake of $\mathcal{K}$ by $W(\mathcal{K})$ and its width by $|W(\mathcal{K})| = w/(d^k - 1)$. Let $\mathcal{A}$ be the hyperbolic component with minimal period, m, in the wake W^q of the $\cdot/q$-satellite component, and assume there is another component $\mathcal{B} \in W^q \setminus W(\mathcal{A})$ with period n. We will show that there are components $\mathcal{A}' \in W^{q+1}$ and $\mathcal{B}' \in W^{q+1} \setminus W(\mathcal{A}')$ with periods $\langle \mathcal{A}' \rangle = k+m$ and $\langle \mathcal{B}' \rangle = k+n$. The claim will then follow from Theorem 6.14, because every visible tree is a subtree of the corresponding formal tree (by 6.4).

Theorem 7.2 now gives the width of the wake W^q; it must contain all the d rays of angle period m which land at $\partial\mathcal{A}$, so

$$\frac{w(d^k-1)}{d^{qk}-1} > \frac{d-1}{d^m-1} \implies$$
$$\frac{w(d^k-1)}{d^{(q+1)k}-1} > \frac{(d-1)(d^{qk}-1)}{(d^m-1)(d^{qk+k}-1)} = \frac{d-1}{d^{k+m}-d^k+d^{k-qk+m}-\dots} > \frac{d-1}{d^{k+m}-1}.$$

Hence there are (at least) $d-1$ rays with angle period $(q-2)k+m =: m'$ in W^q. None of them can have period s strictly dividing m': otherwise W^q would have to contain a sector of width at least $1/(d^s-1) \geq 1/(d^{m'/2}-1)$. But because m was minimal, $|W| = w/(d^k-1) < 1/(d^{m-1}-1)$, so

$$|W^q| = \frac{w(d^k-1)}{d^{qk}-1} < \frac{(d^k-1)^2}{(d^{m-1}-1)(d^{qk}-1)} < \frac{1}{d^{m/2}\cdot d^{(q-2)k/2}} < \frac{1}{d^{m'/2}-1}.$$

Each ray of angle period m' lands at the boundary of some hyperbolic component $\mathcal{A}'$ with the same period, so the wake must contain such a component. Moreover, $\mathcal{A}'$ has exactly d rays landing at its boundary, and the wake must contain them all.

All the rays with angle period n that land at $\partial\mathcal{B}$ must fit in $W^q \setminus W(\mathcal{A})$, so

$$\frac{w(d^k-1)}{d^{qk}-1} - \frac{d-1}{d^m-1} > \frac{u}{d^n-1} \implies$$
$$\frac{w(d^k-1)}{d^{qk+k}-1} - \frac{d-1}{d^{k+m}-1} > \frac{u(d^{qk}-1)}{(d^n-1)(d^{qk+k}-1)} + \frac{(d-1)(d^{qk}-1)}{(d^m-1)(d^{qk+k}-1)} - \frac{d-1}{d^{k+m}-1}$$
$$= \frac{u}{d^{k+n}-d^k+\dots} + (d-1)\cdot\left(\frac{1}{d^{k+m}-d^k+\dots} - \frac{1}{d^{k+m}-1}\right) > \frac{u}{d^{k+n}-1}.$$

The respective domain $W^{q+1} \setminus W(\mathcal{A}')$ between wakes must thus contain at least the same number $u+1$ of rays with angle period $k+n$ as there were those with period n, so it must also contain the component $\mathcal{B}'$ as claimed. □

§8 To Be or Not to Be?

In Sections §5 and §6 we found some reasons for a component to not exist: belonging to a shadow tree or sitting above another nonexistent component. Now we look at some reasons to exist. The special case of a narrow root component is well known. The quadratic case is proved in [L-S-1, 10.2]; we generalize it here for all d.

8.2 Theorem. *If $\mathbf{c} \in \dot{\Sigma}_k^d$ is the kneading sequence of an existent narrow hyperbolic sector $\mathcal{K}$ and $1 \leq n \leq k$, then for every $q = 2, 3, \ldots$ all kneading sequences of the form $\overline{(c_1 \ldots c_k)^{q-1} b_1 b_2 \ldots b_n}$ are realized by hyperbolic components above the q-satellites of $\mathcal{C}$.*

Proof. The number of such kneading sequences is $d^{n-1}(d-1)$ (only $d-1$ choices of b_n define a finite address; see 5.4). The number N of parameter rays with angle denominator $d^{(q-1)k+n} - 1$ in a p/q-subwake of $\mathcal{K}$ must be the multiple of d which is nearest to the product of that denominator with the width of the subwake. By Theorem 7.2, this is

$$\frac{(d^k - 1)(d^{(q-1)k+n} - 1)}{d^{qk} - 1} = d^n - \varepsilon \tag{8.2a}$$

where $0 < \varepsilon \leq 1$, so $N = d^n$. None of these rays has a proper divisor of $(q-1)k+n$ as its exact period, because the p/q-subwake cannot contain components with periods less than $(q-1)k+1$. Hence the number of hyperbolic sectors with period $(q-1)k+n$ is also $(d-1)d^{n-1}$.

To prove that none of these kneading sequences occurs twice, assume there are two components with period $(q-1)k+n$ in the same subwake sharing a kneading sequence. The combinatorial arcs joining them to $\mathcal{K}$ must diverge at a sector $\mathcal{B}$, at internal angles with denominators $r \geq 3$. The minimal period of components in a $\cdot/r$-subwake of $\mathcal{B}$ is at least

$$(r-2)\langle\mathcal{B}\rangle + (q-1)k + 1 > 2(q-1)k + 2 > (q-1)k + n,$$

which is a contradiction. Hence every kneading sequence occurs exactly once. □

Equivalently: for each q, all addresses of the form $C = 1(s_1) \rightarrowtail \cdots \rightarrowtail k(s_k)$ followed by any combination of increasing period numbers $(q-1)k+1, \ldots, qk$ and sector numbers $1, \ldots, d-1$ are realized if C is narrow.

8.3 Corollary. *For all q, the visible q-tree of a narrow k-periodic hyperbolic sector $\mathcal{C}$ consists of k visible components: one of each period $(q-1)k+1, \ldots, qk$. Hence, if $1(s_1) \rightarrowtail \cdots \rightarrowtail k(s_k)$ is the internal address of $\mathcal{C}$, then every address $1(s_1) \rightarrowtail \cdots \rightarrowtail k(s_k) \rightarrowtail n(s_n)$ with $n > k$ is realized.* □

We conclude that for narrow sectors our algorithm 6.4 gives the whole visible tree and nothing more.

Our definition of narrow formal address – one whose periodic word does not begin and end with the same subword of any length (6.3) – is justified by the following lemma:

8.4 Lemma. *Narrow, existent hyperbolic sectors have narrow internal addresses.*

Proof. Let $\mathcal{B} \subset \mathcal{M}^d$ be a narrow sector with period n. If its internal address B were not narrow, the algorithm would yield a formal address A with length less than n above B. Since $\mathcal{B}$ was narrow, A cannot be realized. But by 8.3, VTA only gives realizable addresses, so not even nonexistent components with periods less than n can occur above $\mathcal{B}$. Thus B must be a narrow address. □

Next lemma gives another sufficient condition for existence, independent of the degree d. Figure L illustrates the proof.

8.5 Lemma. *For all d, every formal address with at most four steps is realized by a hyperbolic component in $\mathcal{M}^d$.*

Proof. Consider an address of the form $1(s_1) \rightarrowtail n_2(s_2) \rightarrowtail n_3(s_3) \rightarrowtail n_4$; to simplify notation, we set $n := n_2$. The address $1(s_1) \rightarrowtail n(s_2)$ corresponding to kneading sequence $\overline{\mathbf{s}_1^{n-1}(\mathbf{s}_1+\mathbf{s}_2)}$ is realized by a satellite of the main epicycloid with any numbers $n \in \mathbb{N}$ and $\{s_1, s_2\} \subset \{1, \ldots, d-1\}$; for example, $K^+(1/d^n - 1) = \overline{\mathbf{11}\ldots\mathbf{12}}$. These satellite components are obviously all narrow, so by 8.2 all components with addresses $1(s_1) \rightarrowtail n(s_2) \rightarrowtail n_3$ exist.

To find out which sectors at the next step are narrow, we construct the trees stemming at each sector of the satellite's satellites, $1(s_1) \rightarrowtail n(s_2) \rightarrowtail qn(s_3)$. This address has periodic word $\mathbf{s}_1^{n-1}(\mathbf{s}_1+\mathbf{s}_2)\mathbf{s}_1^{n-1}(\mathbf{s}_1+\mathbf{s}_2+\mathbf{s}_3)$, so it is narrow unless $s_3 = d - s_2$ (for $d = 2$, there is no choice because $s_3 = s_i = 1$ $\forall i$). In this non-narrow case, the word begins and ends with subwords consisting of $\mathbf{s}_1$'s, of lengths $n-1, \ldots, 2, 1$. Thus the formal $\cdot/q$-tree has a fan-like shape, consisting of $n-1$ primitive components with periods $(q-1)n+1, \ldots, qn-1$. All their sectors have periodic words starting with $\mathbf{s}_1^{n-1}(\mathbf{s}_1+\mathbf{s}_2)$, followed by a number of $\mathbf{s}_1$'s, followed by another figure $(\mathbf{s}_1+\mathbf{s}_3) \neq \mathbf{s}_1$, so the primitive components are all narrow.

Theorem 8.2 now implies that all four-step addresses are realized – except, possibly, if $n_3(s_3) = pn(d-s_2)$ for some p. We first consider the case $p = 2$.

All components between the satellite $\mathcal{C} \sim 1(s_1) \rightarrowtail n(s_2) \rightarrowtail 2n(d-s_2)$ and any of the narrow primitive components with period $n+j$ $(1 \leq j < n)$ are found by comparing the kneading sequences as usual (LSA, 4.9). The sequences $\overline{\mathbf{s}_1^{n-1}(\mathbf{s}_1+\mathbf{s}_2)\mathbf{s}_1^{n}} \in \dot{\Sigma}^d_{2n}$ and $\overline{\mathbf{s}_1^{n-1}(\mathbf{s}_1+\mathbf{s}_2)\mathbf{s}_1^{j}} \in \Sigma^d_{n+j,\infty}$ first differ at index $2n+j$. For any j, the next two differences occur at indices $3n$ and $4n$. Hence there do exist $2n$ components with periods $n_4 = n+1, \ldots, 2n-1, 2n+1, \ldots, 3n, 4n$ in $\mathcal{T}_{\cdot/2}(\mathcal{C})$. Using Lemma 6.16, we find all other trees; there are components with periods $n_4 = 3n+1, \ldots, 4n-1, 4n+1, \ldots, 5n, 6n$ in $\mathcal{T}_{\cdot/3}(\mathcal{C})$, and so forth. Thus every address $1(s_1) \rightarrowtail n(s_2) \rightarrowtail 2n(s_3) \rightarrowtail n_4$ is realized.

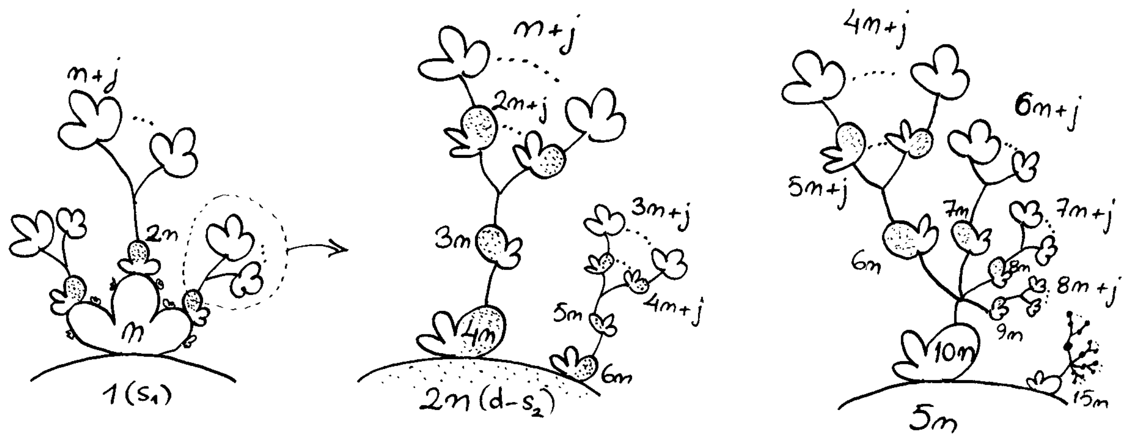

Figure L

The case $p \geq 3$ is more complicated. Each formal tree (grown by VTA, 6.4) of the satellite $\mathcal{B} = 1(s_1) \rightarrowtail n(s_2) \rightarrowtail pn(d-s_2)$ contains a subtree that is combinatorially equivalent to the trees $\mathcal{T}_{\cdot/q}(\mathcal{C})$; the existence of its $n-1$ branch-tips with periods $(p-1)n+j$ is guaranteed by Lemma 7.11. But the formal q-tree also has $q-2$

other branches; tips with periods $(p+i)n+j$ and one with period $(p+1+i)n$ below them ($1 \le i \le q-2$, $1 \le j < n$). Because all components with period equal to that of the base $\mathcal{B}$ (pn) sit above the treetops, their existence can be shown by checking that the sum of their wake-widths is less than $|W(\mathcal{B})| = (d-1)(d^n-1)/(d^{pn}-1)$. The existence of the other components in the formal tree then follows; there are $2(n-1)+1+(q-2)n+1 = qn$ of them, as there should by 6.17. We conclude that the claimed address is indeed realized, even if n_3 is a multiple of $n = n_2$. $\square$

8.6 Remark. This result cannot be improved, because there are nonexistent components with internal addresses of five steps; for example in Λ^2,

$$1 \rightarrowtail 2 \rightarrowtail 4 \rightarrowtail 5 \rightarrowtail 6 \quad \text{(shadow satellite of } A \in \Lambda^2_3)$$
$$1 \rightarrowtail 3 \rightarrowtail 6 \rightarrowtail 7 \rightarrowtail 8 \quad \text{(shadow satellite of } B \in \Lambda^2_4)$$
$$1 \rightarrowtail 2 \rightarrowtail 4 \rightarrowtail 8 \rightarrowtail 9 \quad \text{(in a shadow tree of } C \in \Lambda^2_6)$$

Now we know that the whole tree in Example 6.11 (Figure G) exists: the base sector exists because its internal address $1(1) \rightarrowtail 3(1) \rightarrowtail 6(1) \rightarrowtail 7(1)$ has only four steps. Its periodic word is **1121102**, so the sector is narrow; by 8.3, every address in any of its trees is realized in $\mathcal{M}^3$. This same kneading sequence is also realized for any degree d.

The base component in Example 6.22 (left on Figure J) at internal address $1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 8(2) \rightarrowtail 10$ exists, because its first sector is a subaddress of the infinite root address of the six-periodic component at internal address $A = 1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 6$; $\widehat{\varkappa}_3(A) = \overline{\mathbf{121012}}$, and this gets mapped to

$$1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 8(2) \rightarrowtail 10(1) \rightarrowtail 14(2) \rightarrowtail 16(1) \rightarrowtail 20(2) \rightarrowtail \dots$$

As we saw in previous examples, a formal tree may branch in two ways: either within a component (so the various combinatorial arcs diverge from different sectors) or elsewhere. This latter type means that there are several subwords beginning and ending the periodic word of one sector. Nonexistent components can only occur above this second type of branching-points: if a non-narrow sector $\mathcal{B}$ exists, then there must exist *some* components with periods less than $\langle\mathcal{B}\rangle$ above it. All possible periods are given by VTA. Hence,

8.7 Lemma. *If there is only one number l such that l first and last digits in the periodic word of an existing sector $\mathcal{B}$ coincide, then the component immediately visible over $\mathcal{B}$ with period $\langle\mathcal{B}\rangle - l$ exists too.* $\square$

Lemma 8.7 now guarantees the existence of all components in the second 2-tree in 6.22, and also in the first 2-tree except the ones in the shadow tree of A (with periods 12 and 11).

Now we are ready for one of the main results: even if some component is nonexistent, it "becomes existent" by increasing the degree d and picking right sector numbers.

8.8 Theorem. *Every formal rough address A of length n or less is realized in $\mathcal{M}^d$ by some combination of sector numbers, if $d \ge n/2$.*

We need one more lemma to prove this. The idea is shown in Figure M, where $n_k - k = 9 - 5 = 4$ and $d = 4, 5, 6, 7$.

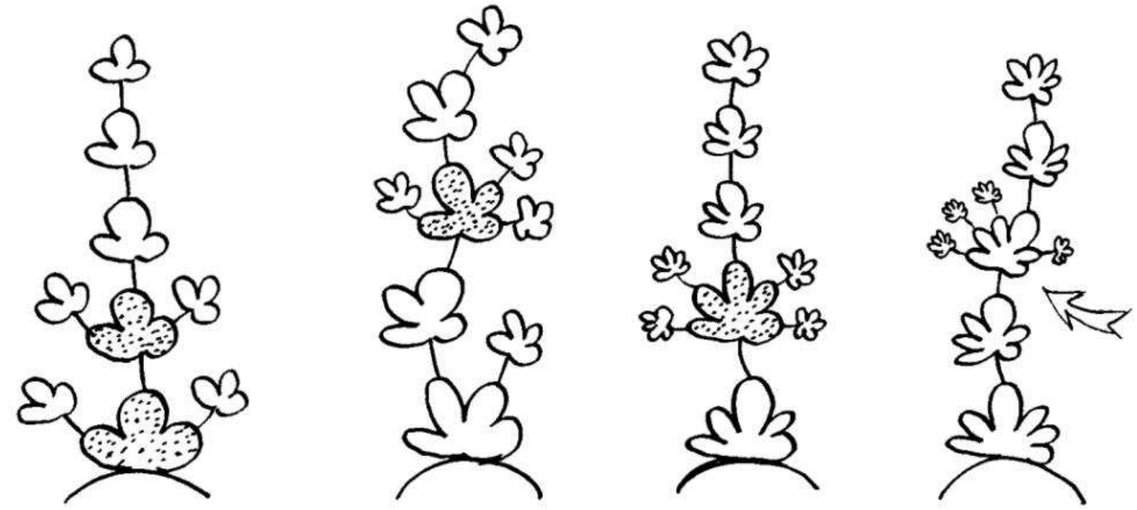

FIGURE M

8.9 Lemma. *If a hyperbolic component $\mathcal{N}$ with internal address $N = 1(s_1) \rightarrowtail \cdots \rightarrowtail n_k(s_k) \rightarrowtail n_{k+1}$ exists in $\mathcal{M}^d$ with $d \geq n_k - k + 3$, then at least one of its sectors is narrow.*

Proof. Each visible tree $\mathcal{T}_{\cdot/q}$ of the sector $\mathcal{K}$ realizing the subaddress $K = 1(s_1) \rightarrowtail \cdots \rightarrowtail n_k(s_k)$ contains (at most) n_k hyperbolic components, by 6.17. One of them is $\mathcal{N}$.

If none of $\mathcal{N}$'s sectors is narrow, then every sector has a component with period less than n_{k+1} behind it. These are all visible from $\mathcal{K}$, because $\mathcal{N}$ is. But then we would have at least d components in the tree, namely $\mathcal{N}$ and one above each of its $d-1$ sectors, so $d \leq n_k$.

We deal with the case $n_k - k + 3 \leq d \leq n_k$ using Lemma 6.6: the formal tree $\widetilde{\mathcal{T}}_{\cdot/q}$ contains a chain of k components (some of which may be nonexistent). Because this secondary trunk can intersect at most one of $\mathcal{N}$'s sector wakes, now actually $k+d-2 \leq n_k$. This contradicts the assumption, so some sector must be narrow. $\square$

Proof of Theorem 8.8. Let $B := 1(s_1) \rightarrowtail n_2(s_2) \rightarrowtail \cdots \rightarrowtail n_k(s_k)$ be the last subaddress of $A = B \rightarrowtail n$. Lemmas 8.3 and 8.9 together make an induction argument to find an existent component of period n_j at each step $j = 1, \ldots, k$.

There is some set of sector numbers s_j such that B is a narrow sector of such a component, as long as $d \geq n_{k-1} - (k-1) + 3 = n_{k-1} - k + 4$. If $k < 3$, then $\mathcal{B}$ is always narrow, so $\mathcal{A}$ exists for every d. Otherwise $k \geq 3$ and $n_k \geq n_{k-1} + 1 \geq n_{k-1} + 1 - k + 3$, so $d \geq n_k$ is a sufficient condition for $\mathcal{A}$ to exist.

By 8.2, now also $B \rightarrowtail ((p-1)n_k + 1)(r_1) \rightarrowtail \cdots \rightarrowtail pn_k(r_{n_k})$ is a realizable address for any integer $p > 1$ and any set of $r_i \in \{0, 1, \ldots, d-1\}$ (here $r_i = 0$ means that the period $(p-1)n_k + i$ does not appear in the address, otherwise r_i is the sector number of the respective subaddress).

When $pn_k \geq n$, the sufficient condition becomes $d \geq n/p$; this is satisfied if $d \geq n/2$. $\square$

8.10 Example. The shortest component address that is nonrealizable in the Mandelbrot set, $1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 5(1) \rightarrowtail 6 \in \Lambda_6^2$, belongs to the shadow two-satellite of the primitive three-periodic component and has kneading sequence $\overline{\mathbf{101100}}$. In Λ_6^3, the same rough address comes in sixteen versions

$\mathbf{12102}*$	$1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 5(1) \rightarrowtail 6$	(purely narrow)
$\mathbf{12100}*$	$1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 5(2) \rightarrowtail 6$	(purely narrow)
$\mathbf{12112}*$	$1(1) \rightarrowtail 2(1) \rightarrowtail 4(2) \rightarrowtail 5(1) \rightarrowtail 6$	(shadow satellite)
$\mathbf{12110}*$	$1(1) \rightarrowtail 2(1) \rightarrowtail 4(2) \rightarrowtail 5(2) \rightarrowtail 6$	

(and the same with $\mathbf{s}_1(\mathbf{s}_1 + \mathbf{s}_2) = \mathbf{10}, \mathbf{20}, \mathbf{21}$). Only four of them have an infinite root address of a lesser-periodic primitive component ($\overline{\mathbf{121}}, \overline{\mathbf{101}}, \overline{\mathbf{202}}, \overline{\mathbf{212}}$) revealing them as shadow satellites. The remaining twelve components are existent, and eight of these are *purely narrow* (one whose all subaddresses are narrow, as defined in [L-S-1]).

Note that (pure) narrowness is not a necessary condition for a continuation of an address to be realizable, and in many cases d may thus be a lot less than $n/2$. Above we discussed the two eleven-periodic components appearing in Example 6.22. Neither sector of the base component $1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 8(2) \rightarrowtail 10$ is narrow; the component at $1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 8(2) \rightarrowtail 10(2) \rightarrowtail 11$ exists while the one at $1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 8(2) \rightarrowtail 10(1) \rightarrowtail 11$ does not.

8.11 Example. All the five rough addresses of length eight that are not realized in $\mathcal{M}_2$, also become existent when $d = 3$. Two of them are above the six-periodic shadow satellite of the previous example, two are shadow satellites of four-periodic sectors, and one is a primitive component in a shadow tree of a five-periodic component. Here are a nonexistent and an existent version of each:

$\mathbf{101100\ 1}*$	$1(1) \rightarrowtail 2(2) \rightarrowtail 4(1) \rightarrowtail 5(2) \rightarrowtail 6(2) \rightarrowtail 8$
$\mathbf{121000\ 1}*$	$1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 5(2) \rightarrowtail 6(2) \rightarrowtail 8$
$\mathbf{101100\ 0}*$	$1(1) \rightarrowtail 2(2) \rightarrowtail 4(2) \rightarrowtail 5(2) \rightarrowtail 6(2) \rightarrowtail 7(2) \rightarrowtail 8$
$\mathbf{121000\ 0}*$	$1(1) \rightarrowtail 2(1) \rightarrowtail 4(1) \rightarrowtail 5(2) \rightarrowtail 6(2) \rightarrowtail 7(2) \rightarrowtail 8$
$\mathbf{1101\ 110}*$	$1(1) \rightarrowtail 3(2) \rightarrowtail 6(1) \rightarrowtail 7(2) \rightarrowtail 8$
$\mathbf{1101\ 120}*$	$1(1) \rightarrowtail 3(2) \rightarrowtail 6(2) \rightarrowtail 7(2) \rightarrowtail 8$
$\mathbf{1001\ 100}*$	$1(1) \rightarrowtail 2(2) \rightarrowtail 3(2) \rightarrowtail 5(1) \rightarrowtail 6(2) \rightarrowtail 7(2) \rightarrowtail 8$
$\mathbf{1001\ 200}*$	$1(1) \rightarrowtail 2(2) \rightarrowtail 3(2) \rightarrowtail 5(2) \rightarrowtail 6(2) \rightarrowtail 7(2) \rightarrowtail 8$
$\mathbf{10111\ 10}*$	$1(1) \rightarrowtail 2(2) \rightarrowtail 4(1) \rightarrowtail 6(1) \rightarrowtail 7(2) \rightarrowtail 8$
$\mathbf{10121\ 20}*$	$1(1) \rightarrowtail 2(2) \rightarrowtail 4(2) \rightarrowtail 6(2) \rightarrowtail 7(2) \rightarrowtail 8$

The next result will narrow our search for shadow components; for each existing sector suspected to hide them behind it, it suffices to look at just one tree, namely the one with denominator 2. (Compare to [Kel, 3.78] which states that among the real trees in case $d = 2$, the 1/2-tree is the only one which may be combinatorially non-equivalent to other trees of the same base component.)

8.12. Theorem. *If a shadow component $\mathcal{F}$ is visible from an existent sector $\mathcal{G}$, then $\mathcal{F}$ sits in the formal $\cdot/2$-tree of $\mathcal{G}$.*

Proof. Because $\mathcal{F}$ is visible from sector $\mathcal{G}$, their periods cannot be equal. If $\langle\mathcal{F}\rangle < \langle\mathcal{G}\rangle$, the claim follows directly from 6.15. Thus, we assume that $\langle\mathcal{F}\rangle > \langle\mathcal{G}\rangle$.

The nonexistent component $\mathcal{F}$ belongs to a shadow tree of some component $\mathcal{B}$; this $\mathcal{B}$ is also visible from $\mathcal{G}$ because $\langle\mathcal{B}\rangle < \langle\mathcal{F}\rangle$, and if there were another component between $\mathcal{G}$ and $\mathcal{B}$ with period less than that, $\mathcal{F}$ would also be above it and thus invisible from $\mathcal{G}$.

Hence (by 6.4) the root sequences $\widehat{\varkappa}(F)$ and $\widehat{\varkappa}(B)$ agree with $\varkappa(G)$ for at least the first $\langle\mathcal{F}\rangle(>\langle\mathcal{G}\rangle)$ figures. The periodic root word of $\mathcal{F}$ is

$$g_1 \dots g_{\langle\mathcal{B}\rangle} \dots g_{\langle\mathcal{F}\rangle} = \begin{cases} (b_1 \dots b'_{\langle\mathcal{B}\rangle})^p & \text{if } p\langle\mathcal{B}\rangle = \langle\mathcal{F}\rangle \\ (b_1 \dots b'_{\langle\mathcal{B}\rangle})^{p-1} b_1 \dots b_r & \text{if } \langle\mathcal{B}\rangle \nmid \langle\mathcal{F}\rangle \end{cases}$$

for some $p \geq 2$, $r = 1, \dots \langle\mathcal{B}\rangle - 1$.

In case $\langle\mathcal{B}\rangle \nmid \langle\mathcal{F}\rangle$ there is a component $\mathcal{A} \succ \mathcal{B}$ such that $\langle\mathcal{A}\rangle = r$, by 6.21. Now $\mathcal{A}, \mathcal{B}, \mathcal{F}$ all belong to the same formal tree $\widetilde{\mathcal{T}}_{./q}(\mathcal{G})$, so their periods must be between $(q-2)\langle\mathcal{G}\rangle$ and $q\langle\mathcal{G}\rangle$, by 6.17. Therefore

$$q\langle\mathcal{G}\rangle > \langle\mathcal{F}\rangle = (p-1)\langle\mathcal{B}\rangle + \langle\mathcal{A}\rangle > 2(q-2)\langle\mathcal{G}\rangle \implies q < 4.$$

In case $\mathcal{F}$ is a shadow satellite of $\mathcal{B}$, $\langle\mathcal{F}\rangle \geq 2\langle\mathcal{B}\rangle$, which leads to the same conclusion. It remains to rule out the possibility that $q = 3$.

If $\mathcal{A}, \mathcal{B}, \mathcal{F} \in \widetilde{\mathcal{T}}_{./3}(\mathcal{G})$, then Lemma 6.14 implies that the tree $\widetilde{\mathcal{T}}_{./2}(\mathcal{G})$ contains components $\mathcal{A}', \mathcal{B}', \mathcal{F}'$ so that $\mathcal{A}' \succ \mathcal{B}'$, and with periods $\langle\mathcal{A}\rangle - \langle\mathcal{G}\rangle$, $\langle\mathcal{B}\rangle - \langle\mathcal{G}\rangle$, $\langle\mathcal{F}\rangle - \langle\mathcal{G}\rangle$, respectively. By 6.19, the shadow tree $\widetilde{\mathcal{V}}_{./p}(\mathcal{B}')$ contains a component $\mathcal{F}''$ with period

$$\langle\mathcal{F}''\rangle = (p-1)\langle\mathcal{B}'\rangle + \langle\mathcal{A}'\rangle = (p-1)\langle\mathcal{B}\rangle + \langle\mathcal{A}\rangle - p\langle\mathcal{G}\rangle = \langle\mathcal{F}'\rangle - (p-1)\langle\mathcal{G}\rangle.$$

(The shadow satellite case is similar, just substitute $\langle\mathcal{B}\rangle$ for $\langle\mathcal{A}\rangle$ in the calculation.) Because this number must be positive and $\langle\mathcal{F}'\rangle < 2\langle\mathcal{G}\rangle$, it follows that $p = 2$ and $\langle\mathcal{F}''\rangle < \langle\mathcal{G}\rangle$. Thus (by 6.12) $\mathcal{F}''$ is visible from $\mathcal{G}$. But then the tree $\widetilde{\mathcal{T}}_{./2}(\mathcal{G})$ contains two components whose periods differ by exactly $\langle\mathcal{G}\rangle$, which contradicts 6.17. Hence necessarily $q = 2$. $\square$

Finally, we show a couple of results concerning the properties of hyperbolic sectors with same kneading sequences but in Mandelbrot sets of different degrees.

8.13. Corollary. *If a kneading sequence* $\mathbf{b} \in \dot{\Sigma}^d_n$ *refers to a shadow sector* $\mathcal{B}$ *of* $\mathcal{M}^d$*, and* $D > d$*, then* $\mathbf{b}$ *is not realized in* $\mathcal{M}^D$ *either.*

Proof. (The proposition makes sense because $\dot{\Sigma}^d_n \subset \dot{\Sigma}^D_n$, by 5.6.)

Assume first that $\mathcal{B}$ is a sector of a shadow satellite, so the sequence $\mathbf{b} = \overline{(c_1 \dots c_k)^{q-1} c_1 \dots (c_k + s)}$ for some divisor k of n and $s \in \{1, 2, \dots, d-1\}$ such that $\overline{c_1 \dots c_k} \subset \Sigma^d_{k,\infty}$. Then $c_{n_i+1} \dots c_k = c_1 \dots c_{k-n_i}$ for the length n_i of the last subaddress of B. This implies that $\overline{c_1 \dots c_k} \subset \Sigma^D_{k,\infty}$ for any degree $D > d$, so $\varkappa_D^{-1}(\mathbf{b}) =: B'$ refers to a shadow satellite sector $\mathcal{B}'$ of $\mathcal{M}^D$.

Otherwise $\mathcal{B}$ sits above some shadow satellite sector of $\mathcal{M}^d$, to which the same argument holds; by Theorem 6.14, $\mathcal{B}'$ is a shadow sector of any Mandelbrot set of higher degree as well. $\square$

8.14. Corollary. *Let kneading sequence* $\overline{c_1 c_2 \dots c_k} \in \dot{\Sigma}_k^d$ *be realized in both* $\mathcal{M}^d$ *and* $\mathcal{M}^{d+1}$. *Then for all* $q \geq 2, n \in \{1, \dots, k\}$, *the component* $\mathcal{B}$ *with kneading sequence*

$$\overline{(c_1 c_2 \dots c_{k-1} \mathbf{d})^{q-1} c_1 \dots c_{n-1} *}$$

is realized in $\mathcal{M}_{d+1}$.

Proof. The kneading sequence $\overline{c_1 c_2 \dots c_{k-1} \mathbf{d}}$ belongs to another sector of the same component $\mathcal{C} \subset \mathcal{M}^{d+1}$ which was assumed to exist. This sector is obviously narrow, because the digit $\mathbf{d}$ does not appear in Σ_k^d and hence no subword can be found at both ends of the periodic word. Now the existence of $\mathcal{B}$ follows from Corollary 8.3. □

8.15. Remarks. It remains open whether all nonexistent components are shadow components for degrees $d > 2$ (if not, the vanishing point of some nonexistent component would not be at the root of some primitive hyperbolic component, but somewhere else in $\mathcal{M}^d$). Examples 8.10 and 8.11 show at least that the nonadmissibility conditions in [B-S] (Remark 5.22 here) are not sufficient for a component to nonexist in a higher-degree Mandelbrot set; the relevant condition must depend on the sector numbers.

If nonexistence of a component *did* imply its being a shadow component, then Theorem 8.12 would imply some stronger results:

- Nonexistent components can only be found in the 1/2-tree of the last subaddress.
- If $B = 1(s_1) \rightarrowtail \cdots \rightarrowtail n_k(s_k)$ is realized and $n_{k+1} \geq 3n_k$, then a hyperbolic component with internal address $B \rightarrowtail n_{k+1}$ exists too.

In 8.14 it would then suffice to assume the sequence to be realized in $\mathcal{M}^d$; existence of a corresponding sector in $\mathcal{M}^{d+1}$ would follow from 8.13 because the argument would work in both directions. Hence, these two results would combine to the following:

- A kneading sequence consisting of d different digits either is realized in all Mandelbrot sets of degree at least d, or in none of them.
- All formal component addresses *not* realizable in $\mathcal{M}^d$ become realized in $\mathcal{M}^{d+1}$ by some sector choice of the last subaddress.

The latter result would improve Theorem 8.8, implying that all rough addresses are realized by some set of sector numbers in $\mathcal{M}^3$.

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