GROWING SYMBOLIC TREES and BUILDING POLYTOPES

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Abstract: *Abstract mathematical concepts, fractals in particular, may carry somewhat unexpected similarities to the natural world. Generalized Mandelbrot set* $\mathcal{M}_d$ *consists of those complex parameters* c *for which the orbit of zero under iteration of polynomial* $f(z) = z^d + c$ *(of degree* $d \geq 2$*) remains bounded. Combinatorial properties, like the tree structure, of these fractal sets can be studied by viewing them as subspaces of a larger, abstract symbol space. Not all such symbolic sequences refer to actual parameters in* $\mathcal{M}_d$*. This gives rise to a visual interpretation of the symbol space: the Mandelbrot set lying flat on the complex plane and "nonexistent" component trees branching off it into another dimension, rather like peculiar shadows. – The connection between symbolic sequences and tree structures also gives an analogy to studying evolutionary trees of living organisms by finding mutations in their DNA sequences. – Visualizing objects with dimensions higher than three may be challenging, but possible. As an example, we construct* 4*-dimensional polytopes using the classical* 3*-dimensional Platonic solids, or regular polyhedra, as building blocks.*

1 FRACTALS and TREES

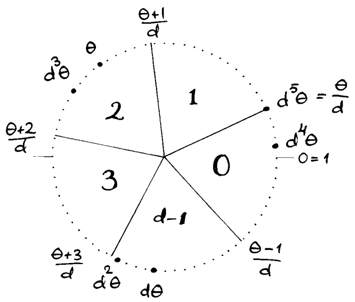
Mandelbrot sets (defined above) are simply connected and compact, infinitely complicated fractal sets on the complex plane with dihedral symmetry groups. Each $\mathcal{M}_d$ contains hyperbolic components, connected by branching, thin threads [4], [8]. We will find an abstract space which turns out to have a natural, similarly tree-like structure.

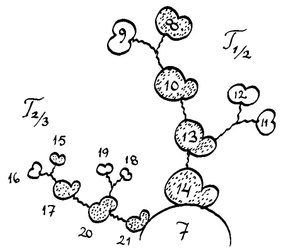
1.1 Abstract Mandelbrot set

The combinatorial properties of polynomials $f : z \mapsto z^d + c$ (and hence also Mandelbrot sets) are based on the fact that to raise a complex number into power of d means to multiply its angle by d (adding c affects little for large $|z|$). Thus we study mappings of angles on the circle [1]; we are particularly interested in angles that are periodic under multiplication by d (modulo full turns). For example,

$$\frac{25}{72} \mapsto \left(\frac{125}{72} = \right) \frac{53}{72} \mapsto \frac{49}{72} \mapsto \frac{29}{72} \mapsto \frac{1}{72} \mapsto \frac{5}{72},$$

so these angles are six-periodic under five-tupling; note that $25/72 = 5425/(5^6 - 1)$.





Angles can be turned into symbolic sequences as follows: given an angle α, divide the circle into d equal sectors at angles $\alpha/d, (\alpha+1)/d, \ldots$ and label them $\mathbf{0}, \mathbf{1}, \ldots, \mathbf{d-1}$ starting from the sector containing angle $0 = 1$. The *kneading sequence* lists the labels of sectors where the iterated angles $\alpha, d\alpha, d^2\alpha, \ldots$ sit. For example, $K(25/72) = \mathbf{24420^1_0 24420^1_0 24420^1_0} \ldots = \overline{\mathbf{24420^1_0}}$.

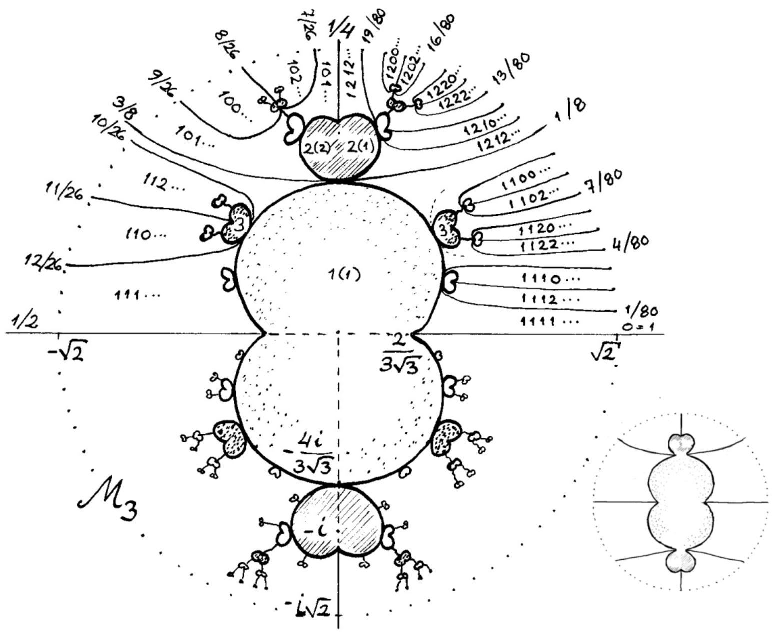
Identifying angles with equal kneading sequences gives rise to a "pinched-disk" model [6]. It consists of "pawprints" with $d-1$ toes, connected by branching threads. For each degree d, this tree-like structure is actually "same" as the Mandelbrot set!

1.2 Symbolic sequence space

Each pawprint in the (abstract) Mandelbrot set has its own kneading sequence. A natural question to ask is, whether the Mandelbrot set contains a "pawprint" (or hyperbolic sector, actually) with any given sequence of appropriate form as its kneading sequence. The answer is no; there are sequences which are not realized. For example, the sequence $\overline{\mathbf{12112}*}$ *non-exists* in $\mathcal{M}_3$ because no angle of the form $a/3^6 - 1 = a/728$ has it as kneading sequence. Hence the abstract Mandelbrot set is a proper subset of a larger symbol space $\mathbf{A}_d$ consisting of all sequences $a_1a_2a_3\ldots$ with $a_1 \neq \mathbf{0}$.

1.3 Growing trees

The definition of kneading sequence implies an important result [5], [7]:



> *When the angle θ moves counter-clockwise around the circle, the nth entry in its kneading sequence changes from j to $j+1$ precisely when θ crosses a rational angle of the form $(rd+j)/(d^n-1)$.*

Given a pair of kneading sequences, A and B, one can thus find the minimal period of angles separating them from each other and from the origin. This information can be used to figure out how the corresponding pawprints are arranged: either the paths leading to each from origin diverge, or one path is contained in the other. This algorithm [7] works for all sequences, realizable or not. Therefore the space $\mathbf{A}_d$ also has a natural tree structure, an extension of the abstract Mandelbrot set.

One can also see from the sequence what other sequences there are "ahead" when looking away from the origin. Given a pawprint $C \sim \overline{c_1 \ldots c_{(n_k)}}$, we find the "visible trees" as follows (however, it may happen that some pawprints obtained this way are nonexistent even though the base C is not):

- for each $q \in \mathbf{N}$ and $s \in \{1, \ldots, d-1\}$, $B \sim \overline{(c_1 \ldots c_{(n_k)})^{q-1} c_1 \ldots (c_{(n_k)} + s)} = \overline{b_1 \ldots b_n}$ corresponds to a satellite of C
- for each B already in the tree, check if $b_1 \ldots b_l = b_{(n-l+1)} \ldots b_n$ for some l. If so, then $A \sim \overline{a_1 \ldots a_m} = \overline{b_1 \ldots (b_{n-l} + r)}$ is above B for all $r \in \{1, \ldots, d-1\}$.

1.4 Evolutionary trees

A somewhat similar method of translating symbolic sequences into tree structures is used by molecular biologists when they reconstruct history of life by studying stretches of DNA molecules or proteins coded by them.

Suppose we have three extant species of animals X, Y, Z, such that the latest common ancestor of two of them has lived more recently than the latest common ancestor of all three of them. Then there are three possible ways they may have evolved. If they have (fictional) DNA sequences

X: ...AAA AAC CCT GTG TGT GTT CGT CGC TCG GTC GTC ATA...
Y: ...AAG AAC CCT GTG TGT GTC CGT CGC TCG GTC GTC ATA...
Z: ...AAG̣ AAC CCT GTG TGT GTC̣ CGT CGC TCG ẠTC GTC ATA...

we see that two mutations separate Y and Z from X, whereas only one mutation separates Z from X and Y. Therefore the hypothesis that the lineage of X branched off earlier – and thus Y and Z are more closely related to each other than either of them is to X – seems more credible than the two alternavive hypotheses.

2 REGULAR POLYTOPES

The first section of this paper dealt with visualizing an abstract mathematical object in two and three dimensions. Polytopes, on the other hand, may in general have any dimension by definition; because the human brain has evolved in a three-dimensional world, visualizing higher dimensions is not easy.

2.1 Regular polytopes of dimensions 2–3

Regular polygon $\{p\}$ can have any number p of vertices. Its corner angle is $\pi(1-\frac{2}{p})$.

Regular polyhedron $\{p,q\}$ has q regular p-gons meeting at each vertex; the midpoints of their edges are the vertices of $\{q\}$. This is possible exactly when $q\pi(1-\frac{2}{p}) < 2\pi$, or $(p-2)(q-2) < 4$, so we have the five Platonic solids

$\{3,3\}$	tetrahedron	$(4,6,4)$				
$\{4,3\}$	cube	$(8,12,6)$	–	$\{5,3\}$	dodekahedron	$(20,30,12)$
$\{3,4\}$	octahedron	$(6,12,8)$	–	$\{3,5\}$	icosahedron	$(12,30,20)$

where (V,E,F) are the numbers of vertices, edges and faces, respectively. The dihedral angle (between the planes of adjacent faces) is $2\arcsin\bigl(\cos\frac{\pi}{q}/\sin\frac{\pi}{p}\bigr)$.

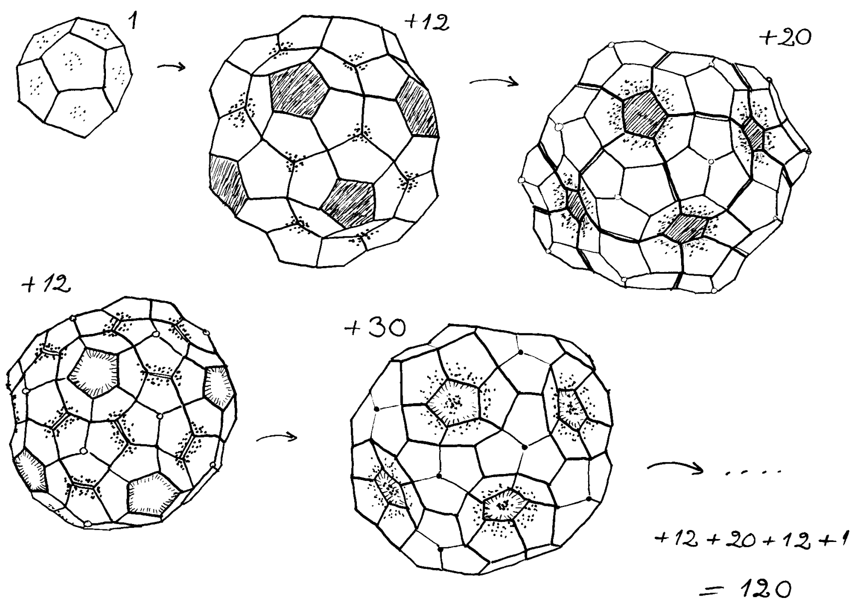
2.2 Regular polytopes of dimension 4

Polytope $\{p,q,r\}$ consists of a number (C) of cells $\{p,q\}$, $r \geq 3$ of them around each edge [2], [3]. Hence r times the dihedral angle must be less than 2π, so $\cos\frac{\pi}{q} < \sin\frac{\pi}{p}\cdot\sin\frac{\pi}{r}$. It follows that there are six possibilities:

$\{3,3,3\}$	simplex	$(5,10,10,5)$	–	$\{3,4,3\}$	$(24,96,96,24)$
$\{4,3,3\}$	hypercube	$(16,32,24,8)$	–	$\{5,3,3\}$	$(600,1200,720,120)$
$\{3,3,4\}$	"co-cube"	$(8,24,32,16)$	–	$\{3,3,5\}$	$(120,720,1200,600)$

The midpoints of all edges meeting at a vertex are the vertices of a regular polyhedron $\{q, r\}$, the vertex figure.

For example, a vertex-tetrahedron can accomodate four cells with $q = 3$. We now construct the polytope $\{5, 3, 3\}$. Starting with one dodekahedron, we first add twelve cells at each face. Then we add a second layer of cells, one into each dent left in between; there are twenty of them. Now the twelve outmost faces of the first layer are still visible; we cover these with a third layer. The fourth layer of cells must be added one edge down, and there are thirty of them. Continuing this way, we see that we need 120 dodekahedra for this polytope; hence its other name, 120-cell.



References

[1] *Atela, P. (1991) Bifurcations of Dynamic Rays in Complex Polynomials of Degree Two,* Ergodic Theory and Dynamical Systems *12, pp. 401–423*

[2] *Berger, M. (1977/1987)* Geometry I–II *(Translation of* Géométrie), *Springer-Verlag*

[3] *Coxeter, H.S.M. (1969)* Introduction to Geometry; *Wiley, New York*

[4] *Carleson, L. and Gamelin, T.W. (1993)* Complex Dynamics; *Universitext, Springer-Verlag*

[5] *Kauko, V. (2000) Trees of Visible Components in the Mandelbrot Set,* Fundamenta Mathematicae *164, pp. 41–60*

[6] *Keller, K. (2000)* Invariant Factors, Julia Equivalences and the (Abstract) Mandelbrot Set; *Lecture Notes in Mathematics #1732, Springer-Verlag*

[7] *Lau, E. and Schleicher, D. (1994) Internal Addresses in the Mandelbrot Set and Irreducibility of Polynomials,* Preprint #19, Institute for Mathematical Sciences, Stony Brook

[8] *Peitgen, H.-O. and Richter, P.H. (1986)* The Beauty of Fractals, *Springer-Verlag*